

# ON THE ASYMPTOTICS OF A TOEPLITZ DETERMINANT WITH SINGULARITIES

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**ABSTRACT.** We provide an alternative proof of the classical single-term asymptotics for Toeplitz determinants whose symbols possess Fisher-Hartwig singularities. We also relax the smoothness conditions on the regular part of the symbols and obtain an estimate for the error term in the asymptotics. Our proof is based on the Riemann-Hilbert analysis of the related systems of orthogonal polynomials and on differential identities for Toeplitz determinants. The result discussed in this paper is crucial for the proof of the asymptotics in the general case of Fisher-Hartwig singularities and extensions to Hankel and Toeplitz+Hankel determinants in [15].

## 1. INTRODUCTION

Let  $f(z)$  be a complex-valued function integrable over the unit circle. Denote its Fourier coefficients

$$f_j = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-ij\theta} d\theta, \quad j = 0, \pm 1, \pm 2, \dots$$

We are interested in the  $n$ -dimensional Toeplitz determinant with symbol  $f(z)$ ,

$$(1.1) \quad D_n(f(z)) = \det(f_{j-k})_{j,k=0}^{n-1}, \quad n \geq 1,$$

where  $f(e^{i\theta})$  has a fixed number of Fisher-Hartwig singularities [20, 27], i.e.,  $f$  has the following form on the unit circle:

$$(1.2) \quad f(z) = e^{V(z)} z^{\sum_{j=0}^m \beta_j} \prod_{j=0}^m |z - z_j|^{2\alpha_j} g_{z_j, \beta_j}(z) z_j^{-\beta_j}, \quad z = e^{i\theta}, \quad \theta \in [0, 2\pi),$$

for some  $m = 0, 1, \dots$ , where

$$(1.3) \quad z_j = e^{i\theta_j}, \quad j = 0, \dots, m, \quad 0 = \theta_0 < \theta_1 < \dots < \theta_m < 2\pi;$$

$$(1.4) \quad g_{z_j, \beta_j}(z) \equiv g_{\beta_j}(z) = \begin{cases} e^{i\pi\beta_j} & 0 \leq \arg z < \theta_j \\ e^{-i\pi\beta_j} & \theta_j \leq \arg z < 2\pi \end{cases},$$

$$(1.5) \quad \Re \alpha_j > -1/2, \quad \beta_j \in \mathbb{C}, \quad j = 0, \dots, m,$$

and  $V(e^{i\theta})$  is a sufficiently smooth function on the unit circle (see below). The condition on the  $\alpha_j$ 's insures integrability. Note that a single Fisher-Hartwig singularity at  $z_j$  consists of a root-type singularity

$$(1.6) \quad |z - z_j|^{2\alpha_j} = \left| 2 \sin \frac{\theta - \theta_j}{2} \right|^{2\alpha_j}$$

and a jump  $e^{i\pi\beta} \rightarrow e^{-i\pi\beta}$ . We assume that  $z_j$ ,  $j = 1, \dots, m$ , are genuine singular points, i.e., either  $\alpha_j \neq 0$  or  $\beta_j \neq 0$ . However, we always include  $z_0 = 1$  explicitly in (1.2), even when  $\alpha_0 = \beta_0 = 0$ : this convention was adopted in [15] in order to facilitate the application of our Toeplitz methods to Hankel determinants. Note that  $g_{\beta_0}(z) = e^{-i\pi\beta_0}$ . Observe that for each  $j \neq 0$ ,  $z^{\beta_j} g_{\beta_j}(z)$  is continuous at  $z = 1$ , and so for each  $j$  each “beta” singularity produces a jump only at the point

$z_j$ . The factors  $z_j^{-\beta_j}$  are singled out to simplify comparisons with the existing literature. Indeed, (1.2) with the notation  $b(\theta) = e^{V(e^{i\theta})}$  is exactly the symbol considered in [20, 3, 4, 8, 9, 10, 18, 19, 29]. However, we write the symbol in a form with  $z^{\sum_{j=0}^m \beta_j}$  factored out. The representation (1.2) is more natural for our analysis.

On the unit circle,  $V(z)$  is represented by its Fourier expansion:

$$(1.7) \quad V(z) = \sum_{k=-\infty}^{\infty} V_k z^k, \quad V_k = \frac{1}{2\pi} \int_0^{2\pi} V(e^{i\theta}) e^{-ki\theta} d\theta.$$

The canonical Wiener-Hopf factorization of  $e^{V(z)}$  is given by

$$(1.8) \quad e^{V(z)} = b_+(z) e^{V_0} b_-(z), \quad b_+(z) = e^{\sum_{k=1}^{\infty} V_k z^k}, \quad b_-(z) = e^{\sum_{k=-\infty}^{-1} V_k z^k}.$$

Define a seminorm

$$(1.9) \quad |||\beta||| = \max_{j,k} |\Re \beta_j - \Re \beta_k|,$$

where the indices  $j, k = 0$  are omitted if  $z = 1$  is not a singular point, i.e. if  $\alpha_0 = \beta_0 = 0$ . If  $m = 0$ , set  $|||\beta||| = 0$ .

In this paper we consider the asymptotics of  $D_n(f)$ ,  $n \rightarrow \infty$ , in the case  $|||\beta||| < 1$ . The case  $|||\beta||| \geq 1$  was addressed in [15]. The asymptotic behavior of  $D_n(f)$  has been studied by many authors (see [15, 19] for a review). An expansion of  $D_n(f)$  for the case  $V \in C^\infty$ ,  $|||\beta||| < 1$ , was obtained by Ehrhardt in [19]. The aim of this paper is to provide an alternative proof of this result based on differential identities for  $D_n(f)$  and a Riemann-Hilbert-Problem analysis of the corresponding system of orthogonal polynomials. We also obtain estimates for the error term and extend the validity of the result to less smooth  $V$ . The analysis of  $D_n(f)$  for  $|||\beta||| < 1$  plays a crucial role in the analysis of  $D_n(f)$  for  $|||\beta||| \geq 1$  (see [15]).

We prove

**Theorem 1.1.** *Let  $f(e^{i\theta})$  be defined in (1.2),  $|||\beta||| < 1$ ,  $\Re \alpha_j > -1/2$ ,  $\alpha_j \pm \beta_j \neq -1, -2, \dots$  for  $j, k = 0, 1, \dots, m$ , and let  $V(z)$  satisfy the condition (1.11), (1.12) below. Then as  $n \rightarrow \infty$ ,*

$$(1.10) \quad D_n(f) = \exp \left[ nV_0 + \sum_{k=1}^{\infty} kV_k V_{-k} \right] \prod_{j=0}^m b_+(z_j)^{-\alpha_j + \beta_j} b_-(z_j)^{-\alpha_j - \beta_j} \\ \times n^{\sum_{j=0}^m (\alpha_j^2 - \beta_j^2)} \prod_{0 \leq j < k \leq m} |z_j - z_k|^{2(\beta_j \beta_k - \alpha_j \alpha_k)} \left( \frac{z_k}{z_j e^{i\pi}} \right)^{\alpha_j \beta_k - \alpha_k \beta_j} \\ \times \prod_{j=0}^m \frac{G(1 + \alpha_j + \beta_j) G(1 + \alpha_j - \beta_j)}{G(1 + 2\alpha_j)} (1 + o(1)),$$

where  $G(x)$  is Barnes'  $G$ -function. The double product over  $j < k$  is set to 1 if  $m = 0$ .

*Remark 1.2.* As indicated above, this result was first obtained by Ehrhardt in the case  $V \in C^\infty$ . We prove the theorem for  $V(z)$  satisfying the smoothness condition

$$(1.11) \quad \sum_{k=-\infty}^{\infty} |k|^s |V_k| < \infty,$$

where

$$(1.12) \quad s > \frac{1 + \sum_{j=0}^m [(\Im \alpha_j)^2 + (\Re \beta_j)^2]}{1 - |||\beta|||}.$$

*Remark 1.3.* In the case of a single singularity, i.e., when  $m = 1$  and  $\alpha_0 = \beta_0 = 0$ , or when  $m = 0$ , the seminorm  $|||\beta||| = 0$ , and the theorem implies that the asymptotic form (1.10) holds for all

$$(1.13) \quad \Re \alpha_m > -\frac{1}{2}, \quad \beta_m \in \mathbb{C}, \quad \alpha_m \pm \beta_m \neq -1, -2, \dots$$

In fact, if there is only one singularity, say  $m = 0$ , and  $V \equiv 0$ , an explicit formula was found by Böttcher and Silbermann [8] for  $D_n(f)$  for any  $n$  in terms of the G-functions:

$$(1.14) \quad D_n(f) = \frac{G(1 + \alpha_0 + \beta_0)G(1 + \alpha_0 - \beta_0)}{G(1 + 2\alpha_0)} \frac{G(n+1)G(n+1+2\alpha_0)}{G(n+1+\alpha_0+\beta_0)G(n+1+\alpha_0-\beta_0)},$$

$$\Re \alpha_0 > -\frac{1}{2}, \quad \alpha_0 \pm \beta_0 \neq -1, -2, \dots, \quad n \geq 1,$$

and (1.10) can then be read off from the known asymptotics of the G-function (see, e.g., [2]).

*Remark 1.4.* Assume that the function  $V(z)$  is sufficiently smooth, i.e. such that  $s$  in (1.11) is, in addition to satisfying (1.12), sufficiently large in comparison with  $|||\beta|||$ . Then we show that the error term  $o(1) = O(n^{|||\beta|||-1})$  in (1.10). In particular, the error term  $o(1) = O(n^{|||\beta|||-1})$  if  $V(z)$  is analytic in a neighborhood of the unit circle. Moreover, for analytic  $V(z)$ , our methods would allow us to calculate, in principle, the full asymptotic expansion rather than just the leading term presented in (1.10). Various regularity properties of the expansion (uniformity, differentiability) in compact sets of parameters satisfying  $\Re \alpha_j > -1/2$ ,  $|||\beta||| < 1$ ,  $\alpha_j \pm \beta_j \neq -1, -2, \dots$ ,  $\theta_j \neq \theta_k$ , are easy to deduce from our analysis.

*Remark 1.5.* Since  $G(-k) = 0$ ,  $k = 0, 1, \dots$ , the formula (1.10) no longer represents the leading asymptotics if  $\alpha_j + \beta_j$  or  $\alpha_j - \beta_j$  is a negative integer for some  $j$ . Although our method applies, we do not address these cases in this paper. It is simply a matter of going deeper in the asymptotic expansion for  $D_n(f)$ . It can happen that  $D_n(f)$  vanishes to all orders (cf. discussion of the Ising model at temperatures above the critical temperature in [16]), and  $e^{-nV_0}D_n(f)$  is exponentially decreasing.

We prove Theorem 1.1 in the following way. We begin by deriving differential identities for the logarithm of  $D_n(f)$  in Section 3, in the spirit of [17, 25, 26, 22, 13, 14], utilizing the polynomials orthogonal with respect to the weight  $f(z)$  on the unit circle. Then, assuming that  $V(z)$  is analytic in a neighborhood of the unit circle, we analyze in Section 4 the asymptotics of these polynomials using Riemann-Hilbert/steepest-descent methods as in [15]. This gives in turn the asymptotics of the differential identities from which the formula (1.10) follows in the  $V \equiv 0$  case by integration w.r.t.  $\alpha_j, \beta_j$  in Section 5.1. However, the error term that results is of order  $n^{2|||\beta|||-1} \ln n$  (see (5.35)), which is asymptotically small only for  $|||\beta||| < 1/2$ , rather than in the full range  $|||\beta||| < 1$ . To prove (1.10) for all  $|||\beta||| < 1$ , we need a finer analysis of cancellations in the Riemann-Hilbert problem as  $n \rightarrow \infty$ . We carry this out in Section 5.2 and reduce the leading order terms in  $D_n(f)$  to a telescopic form (see (5.65)), which leads to a uniform bound on  $D_n(f)n^{\sum_{j=0}^m(\beta_j^2 - \alpha_j^2)}$  for large  $n$  which is valid for all  $|||\beta||| < 1$  and away from the points  $\alpha_j \pm \beta_j = -1, -2, \dots$ . We then apply Vitali's theorem together with the previous result for  $|||\beta||| < 1/2$ . This proves Theorem 1.1 in the  $V \equiv 0$  case as desired, with the error term of order  $n^{|||\beta|||-1}$ . In Sections 5.3, 5.4, we then extend the result to the case of analytic  $V \not\equiv 0$  by applying another differential identity from Section 3.

Ehrhardt proves (1.10) using a “localization” or “separation” technique, introduced by Basor in [4], in which the effect of adding in Fisher-Hartwig singularities one at a time, is controlled. One may also view our approach as a “separation” technique, but in contrast to [4, 19], we add in the Fisher-Hartwig singularities, as well as the regular term  $e^{V(z)}$ , in a continuous fashion.

*Remark 1.6.* An alternative approach to proving Theorem 1.1 in the  $V$ -analytic case is to apply, ab initio, the finer analysis of Section 5.2 to the orthogonal polynomials which appear in the differential identities. This approach is more direct, but is considerably more involved technically. The analysis is resolved, as above, by reduction of the problem to an appropriate telescopic form.

Finally in Section 5.5, we extend our result to the case when  $V(z)$  is not analytic and only satisfies the smoothness condition (1.11), (1.12). We approximate such  $V(z)$  by trigonometric polynomials  $V^{(n)}(z) = \sum_{k=-p}^p V_k z^k$  with an appropriate  $p = p(n)$  and modify the Riemann-Hilbert analysis accordingly. This produces asymptotics of a Toeplitz determinant in whose symbol  $f^{(n)}(z)$  the function  $V(z)$  is replaced by  $V^{(n)}(z)$ . We then use the Heine representation of Toeplitz determinants by multiple integrals to show that  $D_n(f^{(n)})$  approximates  $D_n(f)$  as  $n \rightarrow \infty$ , sufficiently strongly to conclude Theorem 1.1 in the general case.

## 2. RIEMANN-HILBERT PROBLEM

In this section we formulate a Riemann-Hilbert problem (RHP) for the polynomials orthogonal on the unit circle (which, oriented in the positive direction, we denote  $C$ ) with weight  $f(z)$  given by (1.2). We use this RHP in Section 4 to find the asymptotics of the polynomials in the case of analytic  $V(z)$ . Suppose that all  $D_k(f) \neq 0$ ,  $k = k_0, k_0 + 1, \dots$ , for some sufficiently large  $k_0$  (see discussion below). Then the polynomials  $\phi_k(z) = \chi_k z^k + \dots$ ,  $\hat{\phi}_k(z) = \chi_k z^k + \dots$  of degree  $k$ ,  $k = k_0, k_0 + 1, \dots$ , satisfying the orthogonality conditions

$$(2.1) \quad \frac{1}{2\pi} \int_0^{2\pi} \phi_k(z) z^{-j} f(z) d\theta = \chi_k^{-1} \delta_{jk}, \quad \frac{1}{2\pi} \int_0^{2\pi} \hat{\phi}_k(z^{-1}) z^j f(z) d\theta = \chi_k^{-1} \delta_{jk},$$

$$z = e^{i\theta}, \quad j = 0, 1, \dots, k,$$

exist and are given by the following expressions:

$$(2.2) \quad \phi_k(z) = \frac{1}{\sqrt{D_k D_{k+1}}} \begin{vmatrix} f_{00} & f_{01} & \cdots & f_{0k} \\ f_{10} & f_{11} & \cdots & f_{1k} \\ \vdots & \vdots & & \vdots \\ f_{k-10} & f_{k-11} & \cdots & f_{k-1k} \\ 1 & z & \cdots & z^k \end{vmatrix},$$

$$(2.3) \quad \hat{\phi}_k(z^{-1}) = \frac{1}{\sqrt{D_k D_{k+1}}} \begin{vmatrix} f_{00} & f_{01} & \cdots & f_{0k-1} & 1 \\ f_{10} & f_{11} & \cdots & f_{1k-1} & z^{-1} \\ \vdots & \vdots & & \vdots & \vdots \\ f_{k0} & f_{k1} & \cdots & f_{kk-1} & z^{-k} \end{vmatrix},$$

where

$$f_{st} = \frac{1}{2\pi} \int_0^{2\pi} f(z) z^{-(s-t)} d\theta, \quad s, t = 0, 1, \dots, k.$$

We obviously have

$$(2.4) \quad \chi_k = \sqrt{\frac{D_k}{D_{k+1}}}.$$

Consider the following  $2 \times 2$  matrix valued function  $Y^{(k)}(z) \equiv Y(z)$ ,  $k \geq k_0$ :

$$(2.5) \quad Y^{(k)}(z) = \begin{pmatrix} \chi_k^{-1} \phi_k(z) & \chi_k^{-1} \int_C \frac{\phi_k(\xi)}{\xi - z} \frac{f(\xi) d\xi}{2\pi i \xi^k} \\ -\chi_{k-1} z^{k-1} \hat{\phi}_{k-1}(z^{-1}) & -\chi_{k-1} \int_C \frac{\hat{\phi}_{k-1}(\xi^{-1})}{\xi - z} \frac{f(\xi) d\xi}{2\pi i \xi} \end{pmatrix}.$$

It is easy to verify that  $Y(z)$  solves the following Riemann-Hilbert problem:

- (a)  $Y(z)$  is analytic for  $z \in \mathbb{C} \setminus C$ .
- (b) Let  $z \in C \setminus \cup_{j=0}^m z_j$ .  $Y$  has continuous boundary values  $Y_+(z)$  as  $z$  approaches the unit circle from the inside, and  $Y_-(z)$ , from the outside, related by the jump condition

$$(2.6) \quad Y_+(z) = Y_-(z) \begin{pmatrix} 1 & z^{-k} f(z) \\ 0 & 1 \end{pmatrix}, \quad z \in C \setminus \cup_{j=0}^m z_j.$$

- (c)  $Y(z)$  has the following asymptotic behavior at infinity:

$$(2.7) \quad Y(z) = \left( I + O\left(\frac{1}{z}\right) \right) \begin{pmatrix} z^k & 0 \\ 0 & z^{-k} \end{pmatrix}, \quad \text{as } z \rightarrow \infty.$$

- (d) As  $z \rightarrow z_j$ ,  $j = 0, 1, \dots, m$ ,  $z \in \mathbb{C} \setminus C$ ,

$$(2.8) \quad Y(z) = \begin{pmatrix} O(1) & O(1) + O(|z - z_j|^{2\alpha_j}) \\ O(1) & O(1) + O(|z - z_j|^{2\alpha_j}) \end{pmatrix}, \quad \text{if } \alpha_j \neq 0,$$

and

$$(2.9) \quad Y(z) = \begin{pmatrix} O(1) & O(\ln |z - z_j|) \\ O(1) & O(\ln |z - z_j|) \end{pmatrix}, \quad \text{if } \alpha_j = 0, \beta_j \neq 0.$$

(Here and below  $O(a)$  stands for  $O(|a|)$ .) If  $\alpha_0 = \beta_0 = 0$ ,  $Y(z)$  is bounded at  $z = 1$ .

A general fact that orthogonal polynomials can be so represented as a solution of a Riemann-Hilbert problem was noticed in [21] for polynomials on the line and extended to polynomials on the circle in [1]. This fact is important because it turns out that the RHP can be efficiently analyzed for large  $k$  by a steepest-descent type method found in [11] and developed further in many subsequent works. Thus, we first find the solution to the problem (a)–(d) for large  $k$  (applying this method) and then interpret it as the asymptotics of the orthogonal polynomials by (2.5).

Recall the Heine representation for a Toeplitz determinant:

$$(2.10) \quad D_n(f) = \frac{1}{(2\pi)^n n!} \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{1 \leq j < k \leq n} |e^{i\phi_j} - e^{i\phi_k}|^2 \prod_{j=1}^n f(e^{i\phi_j}) d\phi_j.$$

If  $f(z)$  is positive on the unit circle, it follows from (2.10) that  $D_k(f) > 0$ ,  $k = 1, 2, \dots$ , i.e.  $k_0 = 1$ . In the general case, let  $\Lambda$  be a compact subset in the subset  $\{|\beta| < 1, \alpha_j \pm \beta_j \neq -1, -2, \dots\}$  of the parameter space  $\mathcal{P} = \{(\alpha_0, \beta_0, \dots, \alpha_m, \beta_m) : \alpha_j, \beta_j \in \mathbb{C}, \Re \alpha_j > -1/2\}$ . We will show in Section 4 that the RHP (a)–(d) is solvable in  $\Lambda$ , in particular  $\chi_k$  are finite and nonzero, for all sufficiently large  $k$  ( $k \geq k_0(\Lambda)$ ). Let  $\Omega_{k_0}$  be the set of parameters in  $\mathcal{P}$  such that  $D_k(f) = 0$  for some  $k = 1, 2, \dots, k_0 - 1$ . We will then have  $D_k(f) \neq 0$ ,  $k = 1, 2, \dots$  for all points in  $\Lambda \setminus \Omega_{k_0}$ . Note that  $D_n(f)$  depends analytically on  $\alpha_j, \beta_j$  in  $\mathcal{P}$ : this is true, in particular, on (the interior of)  $\Lambda \setminus \Omega_{k_0}$ .

The solution to the RHP (a)–(d) is unique. Note first that  $\det Y(z) = 1$ . Indeed, from the conditions on  $Y(z)$ ,  $\det Y(z)$  is analytic across the unit circle, has all singularities removable, and tends to 1 as  $z \rightarrow \infty$ . It is then identically 1 by Liouville's theorem. Now if there is another solution  $\tilde{Y}(z)$ , we easily obtain by Liouville's theorem that  $\tilde{Y}(z)Y(z)^{-1} \equiv 1$ .

### 3. DIFFERENTIAL IDENTITIES

In this section we derive expressions for the derivative  $(\partial/\partial\gamma) \ln D_n(f(z))$ , where either  $\gamma = \alpha_j$  or  $\gamma = \beta_j$ ,  $j = 0, 1, \dots, m$ , in terms of the matrix elements of (2.5). These will be exact differential identities valid for all  $n = 1, 2, \dots$  (see Proposition 3.1 below), provided all the  $D_n(f) \neq 0$ . We will use these expressions in Section 5.1 to obtain the asymptotics (1.10) in the case  $V \equiv 0$ ,  $\|\beta\| < 1/2$  (improved to  $\|\beta\| < 1$  in Section 5.2). Furthermore, in this section we will derive a differential

identity (see Proposition 3.3 below) which will enable us in Sections 5.3, 5.4 to extend the results to analytic  $V \not\equiv 0$  (improved to sufficiently smooth  $V$  in Section 5.5).

Set  $D_0 \equiv 1$ ,  $\phi_0(z) \equiv \widehat{\phi}_0(z) \equiv 1$ , and suppose that  $D_n(f) \neq 0$  for all  $n = 1, 2, \dots$ . Then the orthogonal polynomials (2.2), (2.3) exist and are analytic in the  $\alpha_j$ 's,  $\beta_j$ 's for all  $k = 1, 2, \dots$ . Moreover, (2.4) implies that

$$(3.1) \quad D_n(f(z)) = \prod_{j=0}^{n-1} \chi_j^{-2}.$$

Note that by orthogonality,

$$(3.2) \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \phi_j(z)}{\partial \gamma} \widehat{\phi}_j(z^{-1}) f(z) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{\partial \chi_j}{\partial \gamma} z^j + \text{polynomial of degree } j-1 \right) \widehat{\phi}_j(z^{-1}) f(z) d\theta = \frac{1}{\chi_j} \frac{\partial \chi_j}{\partial \gamma}.$$

Similarly,

$$(3.3) \quad \frac{1}{2\pi} \int_0^{2\pi} \phi_j(z) \frac{\partial \widehat{\phi}_j(z^{-1})}{\partial \gamma} f(z) d\theta = \frac{1}{\chi_j} \frac{\partial \chi_j}{\partial \gamma}.$$

Therefore, using equation (3.1), we obtain

$$(3.4) \quad \frac{\partial}{\partial \gamma} \ln D_n(f(z)) = \frac{\partial}{\partial \gamma} \ln \prod_{j=0}^{n-1} \chi_j^{-2} = -2 \sum_{j=0}^{n-1} \frac{\partial \chi_j}{\partial \gamma} = -\frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial \gamma} \left( \sum_{j=0}^{n-1} \phi_j(z) \widehat{\phi}_j(z^{-1}) \right) f(z) d\theta.$$

Using here the Christoffel-Darboux identity (see, e.g., Lemma 2.3 of [15])

$$\sum_{k=0}^{n-1} \widehat{\phi}_k(z^{-1}) \phi_k(z) = -n \phi_n(z) \widehat{\phi}_n(z^{-1}) + z \left( \widehat{\phi}_n(z^{-1}) \frac{d}{dz} \phi_n(z) - \phi_n(z) \frac{d}{dz} \widehat{\phi}_n(z^{-1}) \right),$$

and then orthogonality, we can write

$$(3.5) \quad \frac{\partial}{\partial \gamma} \ln D_n(f(z)) = 2n \frac{\partial \chi_n}{\partial \gamma} + \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial \gamma} \left( \phi_n(z) \frac{d \widehat{\phi}_n(z^{-1})}{dz} - \widehat{\phi}_n(z^{-1}) \frac{d \phi_n(z)}{dz} \right) z f(z) d\theta.$$

Writing out the derivative w.r.t.  $\gamma$  in the integral and using orthogonality, we obtain:

$$(3.6) \quad \frac{\partial}{\partial \gamma} \ln D_n(f(z)) = I_1 - I_2,$$

where

$$(3.7) \quad I_1 = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\partial \phi_n(z)}{\partial \gamma} \frac{\partial \widehat{\phi}_n(z^{-1})}{\partial \theta} f(z) d\theta, \quad I_2 = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\partial \phi_n(z)}{\partial \theta} \frac{\partial \widehat{\phi}_n(z^{-1})}{\partial \gamma} f(z) d\theta.$$

It turns out that the particular structure of Fisher-Hartwig singularities allows us to reduce (3.6) to a local formula, i.e. to replace the integrals by the polynomials (and their Cauchy transforms) evaluated only at several points (cf. [22, 26]).

Let us encircle each of the points  $z_j$  by a sufficiently small disc,

$$(3.8) \quad U_{z_j} = \{z : |z - z_j| < \varepsilon\}.$$

Denote

$$(3.9) \quad C_\varepsilon = \cup_{j=0}^m (U_{z_j} \cap C).$$

We now integrate  $I_1$  by parts. First assume that  $V(z) \equiv 0$ . Then, using the expression

$$\frac{\partial f(z)}{\partial \theta} = \sum_{j=0}^m \left( \alpha_j \cot \frac{\theta - \theta_j}{2} + i\beta_j \right) f(z) = \left( \sum_{j=0}^m \alpha_j \frac{z + z_j}{z - z_j} + \beta_j \right) i f(z),$$

we obtain:

$$(3.10) \quad I_1 = -\chi_n^{-1} \frac{\partial \chi_n}{\partial \gamma} \left( n + \sum_{j=0}^m \beta_j \right) - \lim_{\varepsilon \rightarrow 0} \left[ \frac{1}{2\pi} \int_{C \setminus C_\varepsilon} \frac{\partial \phi_n(z)}{\partial \gamma} \hat{\phi}_n(z^{-1}) \left( \sum_{j=0}^m \alpha_j \frac{z + z_j}{z - z_j} \right) f(z) d\theta \right. \\ \left. - \frac{1}{2\pi i} \sum_{j=0}^m \frac{\partial \phi_n(z_j)}{\partial \gamma} \hat{\phi}_n(z_j^{-1}) (f(z_j e^{-i\varepsilon}) - f(z_j e^{i\varepsilon})) \right],$$

where the integration is over  $C \setminus C_\varepsilon$  in the positive direction around the unit circle.

Note that by adding and subtracting  $\frac{\partial \phi_n(z_j)}{\partial \gamma}$ , and by using orthogonality we can write

$$(3.11) \quad \int_{C \setminus C_\varepsilon} \frac{\partial \phi_n(z)}{\partial \gamma} \hat{\phi}_n(z^{-1}) \frac{z + z_j}{z - z_j} f(z) d\theta = \\ \int_{C \setminus C_\varepsilon} \hat{\phi}_n(z^{-1}) \frac{\frac{\partial \phi_n(z)}{\partial \gamma} - \frac{\partial \phi_n(z_j)}{\partial \gamma}}{z - z_j} (z + z_j) f(z) d\theta + \frac{\partial \phi_n(z_j)}{\partial \gamma} \int_{C \setminus C_\varepsilon} \hat{\phi}_n(z^{-1}) \frac{2z_j}{z - z_j} f(z) d\theta + O(\varepsilon^{2\Re \alpha_j + 1}).$$

Obviously, the fraction in the first integral on the r.h.s. is a polynomial in  $z$  of degree  $n - 1$  with leading coefficient  $\partial \chi_n / \partial \gamma$ . Therefore, the integral equals  $2\pi \frac{\partial \chi_n}{\partial \gamma} / \chi_n$  up to  $O(\varepsilon^{2\Re \alpha_j + 1})$ . The second integral can be written in terms of the element  $Y_{22}$  of (2.5) for  $z \rightarrow z_j$ . Let us estimate therefore the following expression for  $\alpha_j \neq 0$ :

$$(3.12) \quad \int_C \hat{\phi}_n(s^{-1}) \frac{2z_j f(s)}{s - z} \frac{ds}{is} - \lim_{\varepsilon \rightarrow 0} \left[ \int_{C \setminus C_\varepsilon} \hat{\phi}_n(s^{-1}) \frac{2z_j f(s)}{s - z_j} \frac{ds}{is} - \frac{1}{i\alpha_j} \hat{\phi}_n(z_j^{-1}) (f(z_j e^{-i\varepsilon}) - f(z_j e^{i\varepsilon})) \right], \\ z \rightarrow z_j, \quad |z| > 1.$$

This difference tends to zero as  $z \rightarrow z_j$ , for  $\Re \alpha_j > 0$ . When  $\Re \alpha_j < 0$ , it is a growing function as  $z \rightarrow z_j$ , and when  $\Re \alpha_j = 0$ ,  $\Im \alpha_j \neq 0$ , an oscillating one. The analysis is similar to that of Section 3 in [26]. For future use, we now fix an analytical continuation of the absolute value, namely, write for  $z$  on the unit circle,

$$(3.13) \quad |z - z_j|^{\alpha_j} = (z - z_j)^{\alpha_j/2} (z^{-1} - z_j^{-1})^{\alpha_j/2} = \frac{(z - z_j)^{\alpha_j}}{(z z_j e^{i\ell_j})^{\alpha_j/2}}, \quad z = e^{i\theta},$$

where  $\ell_j$  is found from the condition that the argument of the above function is zero on the unit circle. Let us fix the cut of  $(z - z_j)^{\alpha_j}$  going along the line  $\theta = \theta_j$  from  $z_j$  to infinity. Fix the branch by the condition that on the line going from  $z_j$  to the right parallel to the real axis,  $\arg(z - z_j) = 2\pi$ . For  $z^{\alpha_j/2}$  in the denominator,  $0 < \arg z < 2\pi$ . If  $z_0 = 1$  let  $0 < \arg(z - 1) < 2\pi$ . (This choice will enable us to use the standard asymptotics for a confluent hypergeometric function in the RH analysis in Section 4 below.) Then, a simple consideration of triangles shows that

$$(3.14) \quad \ell_j = \begin{cases} 3\pi, & 0 < \theta < \theta_j \\ \pi, & \theta_j < \theta < 2\pi \end{cases}$$

Thus (3.13) is continued analytically to neighborhoods of the arcs  $0 < \theta < \theta_j$ , and  $\theta_j < \theta < 2\pi$ . We now analyze (3.12) in the same way that equation (26) was analyzed in [26]. For this analysis, however, we will need two other choices of the function  $(z - z_j)^{2\alpha_j}$ : one choice with the cut going a short distance clockwise along the unit circle  $C$  from  $z_j$ , and another, with the cut going a short distance anticlockwise along  $C$  from  $z_j$ . Let  $c_j$  and  $d_j$  be some points on  $C$  between  $z_j$  and the neighboring singularity in the clockwise and anticlockwise directions, respectively. In a neighborhood of  $z_j$ , let  $g(s)$  be defined by the formula  $\widehat{\phi}_n(s^{-1})f(s)/(is) = |s - z_j|^{2\alpha_j}g(s)$ . We then obtain as in [26] for the part of (3.12) on the arc  $(c_j, d_j)$ :

$$(3.15) \quad \int_{c_j}^{d_j} \frac{|s - z_j|^{2\alpha_j}}{s - z} g(s) ds - \lim_{\varepsilon \rightarrow 0} \left[ \left( \int_{c_j}^{z_j e^{-i\varepsilon}} + \int_{z_j e^{i\varepsilon}}^{d_j} \right) \frac{|s - z_j|^{2\alpha_j}}{s - z} g(s) ds - \frac{\varepsilon^{2\alpha_j}}{2\alpha_j} (g(z_j e^{-i\varepsilon}) - g(z_j e^{i\varepsilon})) \right] \\ = \lim_{\varepsilon \rightarrow 0} \frac{\pi(z - z_j)^{2\alpha_j}}{(zz_j)^{\alpha_j} \sin(2\pi\alpha_j)} (e^{2\pi i\alpha_j - i\alpha\ell_R} g(z_j e^{-i\varepsilon}) - e^{-2\pi i\alpha_j - i\alpha\ell_L} g(z_j e^{i\varepsilon})) \\ + \alpha_j^{-1} O(z - z_j), \quad z \rightarrow z_j, \quad |z| > 1,$$

for  $\alpha_j \neq 0, 1/2, 1, 3/2, \dots$  (for  $\alpha_j = 1/2, 1, 3/2, \dots$  one obtains terms involving  $(z - z_j)^k \ln(z - z_j)$  vanishing as  $z \rightarrow z_j$ ). Here the constants  $\ell_R, \ell_L$  depend on the choice of a branch for  $(z - z_j)^{2\alpha_j}$  (whose cut is, recall, along the circle) and their values will not be important below.

Introduce a “regularized” version of the integral in a neighborhood of  $z_j$ :

$$(3.16) \quad \int_{c_j}^{d_j^{(r)}} \frac{|s - z_j|^{2\alpha_j}}{s - z} g(s) ds \equiv \int_{c_j}^{d_j} \frac{|s - z_j|^{2\alpha_j}}{s - z} g(s) ds \\ - \lim_{\varepsilon \rightarrow 0} \frac{\pi(z - z_j)^{2\alpha_j}}{(zz_j)^{\alpha_j} \sin(2\pi\alpha_j)} (e^{2\pi i\alpha_j - i\alpha\ell_R} g(z_j e^{-i\varepsilon}) - e^{-2\pi i\alpha_j - i\alpha\ell_L} g(z_j e^{i\varepsilon})),$$

for  $z$  in a complex neighborhood of  $z_j$  and  $-1/2 < \Re\alpha_j \leq 0$ ,  $\alpha_j \neq 0$ . If  $\Re\alpha_j > 0$ , we set the “regularized” integral equal to the integral itself.

Denote by  $\widetilde{Y}$  the matrix (2.5), in which the integrals of the second column are replaced by their “regularized” values in a neighborhood of each  $z_j$ .

Then, collecting our observations together, we can write (3.10) in the form

$$(3.17) \quad I_1 = -\chi_n^{-1} \frac{\partial \chi_n}{\partial \gamma} \left( n + \sum_{j=0}^m (\alpha_j + \beta_j) \right) + \sum_{j=0}^m \left\{ \frac{2\alpha_j z_j}{\chi_n} \frac{\partial}{\partial \gamma} \left( \chi_n Y_{11}^{(n)}(z_j) \right) \widetilde{Y}_{22}^{(n+1)}(z_j), \quad \alpha_j \neq 0 \right. \\ \left. \frac{1}{2\pi i} \frac{\partial \widehat{\phi}_n(z_j)}{\partial \gamma} \widehat{\phi}_n(z_j^{-1}) \Delta f(z_j), \quad \alpha_j = 0. \right.$$

where

$$(3.18) \quad \Delta f(z_j) = \lim_{\varepsilon \rightarrow 0} (f(z_j e^{-i\varepsilon}) - f(z_j e^{i\varepsilon})).$$

A similar analysis yields for  $I_2$ :

$$(3.19) \quad I_2 = \chi_n^{-1} \frac{\partial \chi_n}{\partial \gamma} \left( n + \sum_{j=0}^m (\alpha_j - \beta_j) \right) + \sum_{j=0}^m \left\{ 2\chi_n \alpha_j \frac{\partial}{\partial \gamma} \left( \chi_n^{-1} Y_{21}^{(n+1)}(z_j) \right) \widetilde{Y}_{12}^{(n)}(z_j), \quad \alpha_j \neq 0 \right. \\ \left. \frac{1}{2\pi i} \frac{\partial \widehat{\phi}_n(z_j^{-1})}{\partial \gamma} \phi_n(z_j) \Delta f(z_j), \quad \alpha_j = 0. \right.$$

Substituting these results into (3.6) we obtain



**Proposition 3.1.** *Let  $V(z) \equiv 0$ . Let  $\gamma = \alpha_k$  or  $\gamma = \beta_k$ ,  $k = 0, 1, \dots, m$ , and  $D_n(f(z)) \neq 0$  for all  $n$ . Then for any  $n = 1, 2, \dots$ ,*

$$(3.20) \quad \frac{\partial}{\partial \gamma} \ln D_n(f(z)) = -2\chi_n^{-1} \frac{\partial \chi_n}{\partial \gamma} \left( n + \sum_{j=0}^m \alpha_j \right) \\ + \sum_{j=0}^m \left\{ 2\alpha_j \left\{ \frac{\partial}{\partial \gamma} \left( \chi_n Y_{11}^{(n)}(z_j) \right) z_j \chi_n^{-1} \tilde{Y}_{22}^{(n+1)}(z_j) - \frac{\partial}{\partial \gamma} \left( \chi_n^{-1} Y_{21}^{(n+1)}(z_j) \right) \chi_n \tilde{Y}_{12}^{(n)}(z_j) \right\}, \quad \alpha_j \neq 0 \right. \\ \left. + \frac{1}{2\pi i} \left\{ \frac{\partial \phi_n(z_j)}{\partial \gamma} \hat{\phi}_n(z_j^{-1}) - \frac{\partial \hat{\phi}_n(z_j^{-1})}{\partial \gamma} \phi_n(z_j) \right\} \Delta f(z_j), \quad \alpha_j = 0, \right.$$

where  $\Delta f(z_j)$  is defined in (3.18).

In Section 5.1 we substitute the asymptotics for  $Y$  (found in Section 4) in (3.20) and, by integrating, obtain part of Theorem 1.1 for  $f(z)$  with  $V(z) \equiv 0$ ,  $|||\beta||| < 1/2$ . Further analysis of Section 5.2 extends the result to  $|||\beta||| < 1$ .

*Remark 3.2.* The differential identities (3.20) admit an interesting interpretation in the context of the monodromy theory of the Fuchsian system of linear ODEs canonically related to the Riemann-Hilbert problem (2.6)–(2.9). We explain this connection in some detail in the Appendix. The results presented in the Appendix, however, are not used in the main body of the paper.

To extend the theorem to nonzero  $V(z)$  we will use another differential identity. Let us introduce a parametric family of weights and the corresponding orthogonal polynomials indexed by  $t \in [0, 1]$ . Namely, let

$$(3.21) \quad f(z, t) = (1 - t + te^{V(z)})e^{-V(z)}f(z).$$

Thus  $f(z, 0)$  corresponds to  $f(z)$  with  $V = 0$ , whereas  $f(z, 1)$  gives the function (1.2) we are interested in.

Note that

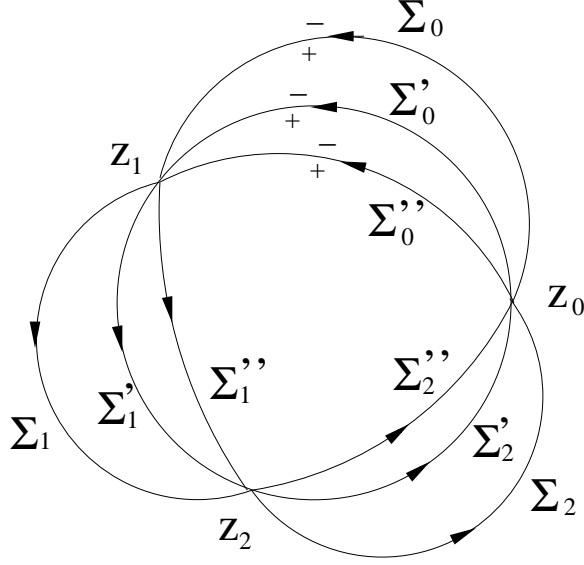
$$(3.22) \quad \frac{\partial f(z, t)}{\partial t} = \frac{f(z, t) - f(z, 0)}{t}.$$

Set now  $\gamma = t$  and replace the function  $f(z)$  and the orthogonal polynomials in (3.5) by  $f(z, t)$  and the polynomials orthogonal w.r.t.  $f(z, t)$ . Then the integral in the r.h.s. of (3.5) can be written as follows (we assume  $D_n \neq 0$  for all  $n$ ):

$$(3.23) \quad \frac{\partial}{\partial t} \left[ \frac{1}{2\pi} \int_0^{2\pi} \left( \phi_n(z, t) \frac{d\hat{\phi}_n(z^{-1}, t)}{dz} - \hat{\phi}_n(z^{-1}, t) \frac{d\phi_n(z, t)}{dz} \right) z f(z, t) d\theta \right] \\ - \frac{1}{2\pi} \int_0^{2\pi} \left( \phi_n(z, t) \frac{d\hat{\phi}_n(z^{-1}, t)}{dz} - \hat{\phi}_n(z^{-1}, t) \frac{d\phi_n(z, t)}{dz} \right) z \frac{f(z, t) - f(z, 0)}{t} d\theta \\ = \frac{2n}{t} + \frac{1}{2\pi t} \int_C \left( \phi_n(z, t) \frac{d\hat{\phi}_n(z^{-1}, t)}{dz} - \hat{\phi}_n(z^{-1}, t) \frac{d\phi_n(z, t)}{dz} \right) z f(z, 0) dz.$$

Therefore, we obtain

$$(3.24) \quad \frac{\partial}{\partial t} \ln D_n(f(z, t)) = 2n \left( \frac{1}{t} + \chi_n^{-1} \frac{\partial \chi_n}{\partial t} \right) \\ + \frac{1}{2\pi t} \int_C \left( \phi_n(z, t) \frac{d\hat{\phi}_n(z^{-1}, t)}{dz} - \hat{\phi}_n(z^{-1}, t) \frac{d\phi_n(z, t)}{dz} \right) z f(z, 0) dz.$$

FIGURE 1. Contour for the  $S$ -Riemann-Hilbert problem ( $m = 2$ ).

To write this identity in terms of the solution to the RHP (2.6) - (2.7), note first that using the recurrence relation (see, e.g., Lemma 2.2 of [15])

$$\chi_n z^{-1} \hat{\phi}_n(z^{-1}) = \chi_{n+1} \hat{\phi}_{n+1}(z^{-1}) - \hat{\phi}_{n+1}(0) z^{-n-1} \phi_{n+1}(z),$$

we have

$$(3.25) \quad Y_{21}(z, t) = -\chi_{n-1} z^{n-1} \hat{\phi}_{n-1}(z^{-1}, t) = -\chi_n z^n \hat{\phi}_n(z^{-1}, t) + \hat{\phi}_n(0, t) \phi_n(z, t).$$

Now using the orthogonality relations (2.1) and the formulae (3.2) and (3.3), we obtain from (3.24)

**Proposition 3.3.** *Let  $f(z, t)$  be given by (3.21) and  $D_n(f(z, t)) \neq 0$  for all  $n$ . Let  $\phi_k(z, t)$ ,  $\hat{\phi}_k(z, t)$ ,  $k = 0, 1, \dots$ , be the corresponding orthogonal polynomials. Then for any  $n = 1, 2, \dots$ ,*

$$(3.26) \quad \frac{\partial}{\partial t} \ln D_n(f(z, t)) = \frac{1}{2\pi i} \int_C z^{-n} \left( Y_{11}(z, t) \frac{\partial Y_{21}(z, t)}{\partial z} - Y_{21}(z, t) \frac{\partial Y_{11}(z, t)}{\partial z} \right) \frac{\partial f(z, t)}{\partial t} dz,$$

where the integration is over the unit circle.

#### 4. ASYMPTOTICS FOR THE RIEMANN-HILBERT PROBLEM

The RHP of Section 2 was solved in [15]. In this section we list the results from [15] we need below for the proof of Theorem 1.1. We always assume (for the rest of the paper) that  $f(z)$  is given by (1.2) and that  $\alpha_j \pm \beta_j \neq -1, -2, \dots$  for all  $j = 0, 1, \dots, m$ . In this section 4 we also assume for simplicity that  $z_0 = 1$  is a singularity. However, the results trivially extend to the case  $\alpha_0 = \beta_0 = 0$ . In this section 4, we further assume that  $V(z)$  is analytic in a neighborhood of the unit circle.

First, set

$$(4.1) \quad T(z) = Y(z) \begin{cases} z^{-n\sigma_3}, & |z| > 1 \\ I, & |z| < 1. \end{cases}$$

Now split the contour as shown in Figure 1. Set

$$(4.2) \quad S(z) = \begin{cases} T(z), & \text{for } z \text{ outside the lenses,} \\ T(z) \begin{pmatrix} 1 & 0 \\ f(z)^{-1} z^{-n} & 1 \end{pmatrix}, & \text{for } |z| > 1 \text{ and inside the lenses,} \\ T(z) \begin{pmatrix} 1 & 0 \\ -f(z)^{-1} z^n & 1 \end{pmatrix}, & \text{for } |z| < 1 \text{ and inside the lenses.} \end{cases}$$

Here  $f(z)$  is the analytic continuation of  $f(z)$  off the unit circle into the inside of the lenses as discussed following (3.13).

The function  $S(z)$  satisfies the following Riemann Hilbert problem:

- (a)  $S(z)$  is analytic for  $z \in \mathbb{C} \setminus \Sigma$ , where  $\Sigma = \cup_{j=0}^m (\Sigma_j \cup \Sigma'_j \cup \Sigma''_j)$ .
- (b) The boundary values of  $S(z)$  are related by the jump condition

$$(4.3) \quad S_+(z) = S_-(z) \begin{pmatrix} 1 & 0 \\ f(z)^{-1} z^{\mp n} & 1 \end{pmatrix}, \quad z \in \cup_{j=0}^m (\Sigma_j \cup \Sigma''_j),$$

where the minus sign in the exponent is on  $\Sigma_j$ , and plus on  $\Sigma''_j$ ,

$$(4.4) \quad S_+(z) = S_-(z) \begin{pmatrix} 0 & f(z) \\ -f(z)^{-1} & 0 \end{pmatrix}, \quad z \in \cup_{j=0}^m \Sigma'_j.$$

- (c)  $S(z) = I + O(1/z)$  as  $z \rightarrow \infty$ ,
- (d) As  $z \rightarrow z_j$ ,  $j = 0, \dots, m$ ,  $z \in \mathbb{C} \setminus C$  outside the lenses,

$$(4.5) \quad S(z) = \begin{pmatrix} O(1) & O(1) + O(|z - z_j|^{2\alpha_j}) \\ O(1) & O(1) + O(|z - z_j|^{2\alpha_j}) \end{pmatrix}$$

if  $\alpha_j \neq 0$ , and

$$(4.6) \quad S(z) = \begin{pmatrix} O(1) & O(\ln |z - z_j|) \\ O(1) & O(\ln |z - z_j|) \end{pmatrix}$$

if  $\alpha_j = 0$ ,  $\beta_j \neq 0$ . The behavior of  $S(z)$  for  $z \rightarrow z_j$  in other sectors is obtained from these expressions by application of the appropriate jump conditions.

We now present formulae for the parametrices which solve the model Riemann Hilbert problems outside the neighborhoods  $U_{z_j}$  of the points  $z_j$ , and inside those neighborhoods, respectively. These parametrices match to the leading order in  $n$  on the boundaries of the neighborhoods  $U_{z_j}$ , and this matching allows us to construct the asymptotic solution to the RHP for  $Y$ .

The parametrix outside the  $U_{z_j}$ 's is the following:

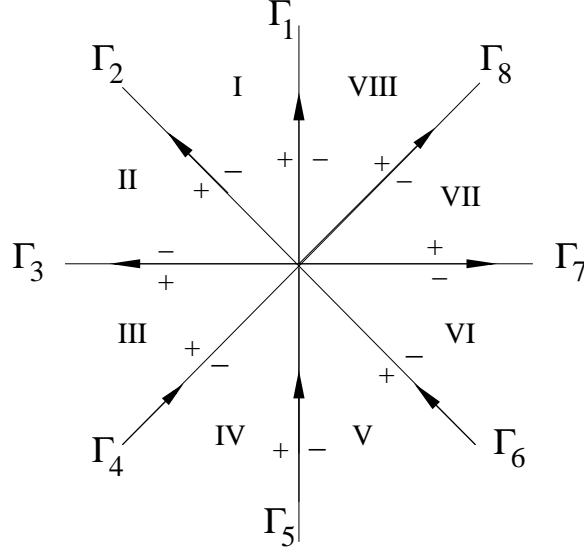
$$(4.7) \quad N(z) = \begin{cases} \mathcal{D}(z)^{\sigma_3}, & |z| > 1 \\ \mathcal{D}(z)^{\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & |z| < 1 \end{cases}, \quad z \in \mathbb{C} \setminus \cup_{j=0}^m U_{z_j}$$

where the Szegő function

$$(4.8) \quad \mathcal{D}(z) = \exp \frac{1}{2\pi i} \int_C \frac{\ln f(s)}{s - z} ds,$$

is analytic away from the unit circle, and we have

$$(4.9) \quad \mathcal{D}(z) = e^{V_0} b_+(z) \prod_{k=0}^m \left( \frac{z - z_k}{z_k e^{i\pi}} \right)^{\alpha_k + \beta_k}, \quad |z| < 1.$$

FIGURE 2. The auxiliary contour for the parametrix at  $z_j$ .

and

$$(4.10) \quad \mathcal{D}(z) = b_-(z)^{-1} \prod_{k=0}^m \left( \frac{z - z_k}{z} \right)^{-\alpha_k + \beta_k}, \quad |z| > 1,$$

where  $V_0, b_{\pm}(z)$  are defined in (1.8). Note that the branch of  $(z - z_k)^{\pm\alpha_k + \beta_k}$  in (4.9, 4.10) is taken as discussed following equation (3.13) above. In (4.10) for any  $k$ , the cut of the root  $z^{-\alpha_k + \beta_k}$  is the line  $\theta = \theta_k$  from  $z = 0$  to infinity, and  $\theta_k < \arg z < 2\pi + \theta_k$ .

Inside each neighborhood  $U_{z_j}$  the parametrix is given in terms of a confluent hypergeometric function. First, set

$$(4.11) \quad \zeta = n \ln \frac{z}{z_j},$$

where  $\ln x > 0$  for  $x > 1$ , and has a cut on the negative half of the real axis. Under this transformation the neighborhood  $U_{z_j}$  is mapped into a neighborhood of zero in the  $\zeta$ -plane. Note that the transformation  $\zeta(z)$  is analytic, one-to-one, and it takes an arc of the unit circle to an interval of the imaginary axis. Let us now choose the exact form of the cuts  $\Sigma$  in  $U_{z_j}$  so that their images under the mapping  $\zeta(z)$  are straight lines (Figure 2).

We add one more jump contour to  $\Sigma$  in  $U_{z_j}$  which is the pre-image of the real line  $\Gamma_3$  and  $\Gamma_7$  in the  $\zeta$ -plane. This is needed below because of the non-analyticity of the function  $|z - z_j|^{\alpha_j}$ . Note that we can construct two different analytic continuations of this function off the unit circle to the pre-images of the upper and lower half  $\zeta$ -plane, respectively. Namely, let

$$(4.12) \quad h_{\alpha_j}(z) = |z - z_j|^{\alpha_j}, \quad z = e^{i\theta}$$

with the branches chosen as in (3.13). As remarked above, (3.13) is continued analytically to neighborhoods of the arcs  $0 < \theta < \theta_j$ , and  $\theta_j < \theta < 2\pi$ . In  $U_{z_j}$ , we extend these neighborhoods to the pre-images of the lower and upper half  $\zeta$ -plane (intersected with  $\zeta(U_{z_j})$ ), respectively. The cut of  $h_{\alpha_j}$  is along the contours  $\Gamma_3$  and  $\Gamma_7$  in the  $\zeta$ -plane.

For  $z \rightarrow z_j$ ,  $\zeta = n(z - z_j)/z_j + O((z - z_j)^2)$ . We have  $0 < \arg \zeta < 2\pi$ , which follows from the choice of  $\arg(z - z_j)$  in (3.13).

From now on we will provide the formulae for the parametrix only in the region  $I$  (see [15] for the complete results). Set

$$(4.13) \quad F_j(z) = e^{\frac{V(z)}{2}} \prod_{k=0}^m \left( \frac{z}{z_k} \right)^{\beta_k/2} \prod_{k \neq j} h_{\alpha_k}(z) g_{\beta_k}(z)^{1/2} h_{\alpha_j}(z) e^{-i\pi\alpha_j}, \quad \zeta \in I, \quad z \in U_{z_j}, \quad j \neq 0.$$

Note that this function is related to  $f(z)$  as follows:

$$(4.14) \quad F_j(z)^2 = f(z) e^{-2\pi i\alpha_j} g_{\beta_j}^{-1}(z) \quad \zeta \in I.$$

The functions  $g_{\beta_k}(z)$  are defined in (1.4). The formulae for  $F_0(z)$  are the same, but with  $g_{\beta_0}(z)$  replaced with

$$(4.15) \quad \widehat{g}_{\beta_0}(z) = \begin{cases} e^{-i\pi\beta_0}, & \arg z > 0 \\ e^{i\pi\beta_0}, & \arg z < 2\pi \end{cases}, \quad z \in U_{z_0}.$$

We then have the following expression for the parametrix  $P_j(z)$  in the region  $z(I)$  of  $U_{z_j}$ :

$$(4.16) \quad P_{z_j}(z) = E(z) \Psi_j(\zeta) F_j(z)^{-\sigma_3} z^{n\sigma_3/2}, \quad \zeta \in I.$$

Here

$$(4.17) \quad E(z) = N(z) \zeta^{\beta_j\sigma_3} F_j^{\sigma_3}(z) z_j^{-n\sigma_3/2} \begin{pmatrix} e^{-i\pi(2\beta_j+\alpha_j)} & 0 \\ 0 & e^{i\pi(\beta_j+2\alpha_j)} \end{pmatrix}$$

and

$$(4.18) \quad \Psi_j(\zeta) = \begin{pmatrix} \zeta^{\alpha_j} \psi(\alpha_j + \beta_j, 1 + 2\alpha_j, \zeta) e^{i\pi(2\beta_j+\alpha_j)} e^{-\zeta/2} & \\ -\zeta^{-\alpha_j} \psi(1 - \alpha_j + \beta_j, 1 - 2\alpha_j, \zeta) e^{i\pi(\beta_j-3\alpha_j)} e^{-\zeta/2} \frac{\Gamma(1+\alpha_j+\beta_j)}{\Gamma(\alpha_j-\beta_j)} & \\ & -\zeta^{\alpha_j} \psi(1 + \alpha_j - \beta_j, 1 + 2\alpha_j, e^{-i\pi}\zeta) e^{i\pi(\beta_j+\alpha_j)} e^{\zeta/2} \frac{\Gamma(1+\alpha_j-\beta_j)}{\Gamma(\alpha_j+\beta_j)} \\ & \zeta^{-\alpha_j} \psi(-\alpha_j - \beta_j, 1 - 2\alpha_j, e^{-i\pi}\zeta) e^{-i\pi\alpha_j} e^{\zeta/2} \end{pmatrix},$$

where  $\psi(a, b, x)$  is the confluent hypergeometric function of the second kind, and  $\Gamma(x)$  is Euler's  $\Gamma$ -function. Recall our assumption that  $\alpha_j \pm \beta_j \neq -1, -2, \dots$

The matching condition for the parametrices  $P_{z_j}$  and  $N$  is the following for any  $k = 1, 2, \dots$ :

$$(4.19) \quad P_{z_j}(z) N^{-1}(z) = I + \Delta_1(z) + \Delta_2(z) + \dots + \Delta_k(z) + \Delta_{k+1}^{(r)}, \quad z \in \partial U_{z_j}.$$

Every  $\Delta_p(z)$ ,  $\Delta_p^{(r)}(z)$ ,  $p = 1, 2, \dots$ ,  $z \in \partial U_{z_j}$  is of the form

$$(4.20) \quad a_j^{-\sigma_3} O(n^{-p}) a_j^{\sigma_3}, \quad a_j \equiv n^{\beta_j} z_j^{-n/2}.$$

In particular, explicitly, on the part of  $\partial U_{z_j}$  whose  $\zeta$ -image is in  $I$ ,

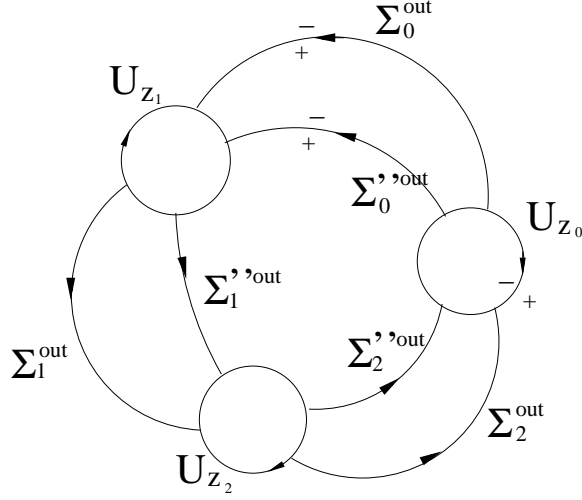
$$(4.21) \quad \Delta_1(z) = \frac{1}{\zeta} \begin{pmatrix} -(\alpha_j^2 - \beta_j^2) & \frac{\Gamma(1+\alpha_j+\beta_j)}{\Gamma(\alpha_j-\beta_j)} \left( \frac{\mathcal{D}(z)}{\zeta^{\beta_j} F_j(z)} \right)^2 z_j^n e^{i\pi(2\beta_j-\alpha_j)} \\ -\frac{\Gamma(1+\alpha_j-\beta_j)}{\Gamma(\alpha_j+\beta_j)} \left( \frac{\mathcal{D}(z)}{\zeta^{\beta_j} F_j(z)} \right)^{-2} z_j^{-n} e^{-i\pi(2\beta_j-\alpha_j)} & \alpha_j^2 - \beta_j^2 \end{pmatrix},$$

which extends to a meromorphic function in a neighborhood of  $U_{z_j}$  with a simple pole at  $z = z_j$ .

The error term  $\Delta_{k+1}^{(r)}$  in (4.19) is uniform in  $z$  on  $\partial U_{z_j}$ .

At the point  $z_j$  we have

$$(4.22) \quad F_j(z) = \eta_j e^{-3i\pi\alpha_j/2} z_j^{-\alpha_j} u^{\alpha_j} (1 + O(u)), \quad u = z - z_j, \quad \zeta \in I,$$

FIGURE 3. Contour  $\Gamma$  for the  $R$  and  $\tilde{R}$  Riemann-Hilbert problems ( $m = 2$ ).

where

$$(4.23) \quad \eta_j = e^{V(z_j)/2} \exp \left\{ -\frac{i\pi}{2} \left( \sum_{k=0}^{j-1} \beta_k - \sum_{k=j+1}^m \beta_k \right) \right\} \prod_{k \neq j} \left( \frac{z_j}{z_k} \right)^{\beta_k/2} |z_j - z_k|^{\alpha_k},$$

and

$$(4.24) \quad \left( \frac{\mathcal{D}(z)}{\zeta^{\beta_j} F_j(z)} \right)^2 = \mu_j^2 e^{i\pi(\alpha_j - 2\beta_j)} n^{-2\beta_j} (1 + O(u)), \quad u = z - z_j, \quad \zeta \in I,$$

$$(4.25) \quad \mu_j = \left( e^{V_0} \frac{b_+(z_j)}{b_-(z_j)} \right)^{1/2} \exp \left\{ -\frac{i\pi}{2} \left( \sum_{k=0}^{j-1} \alpha_k - \sum_{k=j+1}^m \alpha_k \right) \right\} \prod_{k \neq j} \left( \frac{z_j}{z_k} \right)^{\alpha_k/2} |z_j - z_k|^{\beta_k}.$$

The sums from 0 to  $-1$  for  $j = 0$  and from  $m + 1$  to  $m$  for  $j = m$  are set to zero.

**4.1. R-RHP.** Let

$$(4.26) \quad R(z) = \begin{cases} S(z)N^{-1}(z), & z \in U_\infty \setminus \Gamma, \\ S(z)P_{z_j}^{-1}(z), & z \in U_{z_j} \setminus \Gamma, \end{cases} \quad \begin{aligned} U_\infty &= \mathbb{C} \setminus \cup_{j=0}^m U_{z_j}, \\ j &= 0, \dots, m. \end{aligned}$$

It is easy to verify that this function has jumps only on  $\partial U_{z_j}$ , and the parts of  $\Sigma_j, \Sigma_j''$  lying outside the neighborhoods  $U_{z_j}$  (we denote these parts without the end-points  $\Sigma_j^{\text{out}}, \Sigma_j''^{\text{out}}$ ). The full contour  $\Gamma$  is shown in Figure 3. Away from  $\Gamma$ , as a standard argument shows,  $R(z)$  is analytic. Moreover, we have:  $R(z) = I + O(1/z)$  as  $z \rightarrow \infty$ .

The jumps of  $R(z)$  are as follows:

$$(4.27) \quad R_+(z) = R_-(z)N(z) \begin{pmatrix} 1 & 0 \\ f(z)^{-1}z^{-n} & 1 \end{pmatrix} N(z)^{-1}, \quad z \in \Sigma_j^{\text{out}},$$

$$(4.28) \quad R_+(z) = R_-(z)N(z) \begin{pmatrix} 1 & 0 \\ f(z)^{-1}z^n & 1 \end{pmatrix} N(z)^{-1}, \quad z \in \Sigma_j''^{\text{out}},$$

$$(4.29) \quad R_+(z) = R_-(z)P_{z_j}(z)N(z)^{-1}, \quad z \in \partial U_{z_j} \setminus \{ \text{intersection points} \}, \\ j = 0, \dots, m.$$

The jump matrix on  $\Sigma^{\text{out}}, \Sigma''^{\text{out}}$  can be estimated uniformly in  $\alpha_j, \beta_j$  as  $I + O(\exp(-\varepsilon n))$ , where  $\varepsilon$  is a positive constant. The jump matrices on  $\partial U_{z_j}$  admit a uniform expansion (4.19) in inverse powers of  $n$  conjugated by  $n^{\beta_j \sigma_3} z_j^{-n \sigma_3 / 2}$ , and (4.20) is of order  $n^{2 \max_j |\Re \beta_j| - p}$ . To obtain the standard solution of the  $R$ -RHP in terms of a Neuman series (see, e.g., [12]) we must have  $n^{2 \max_j |\Re \beta_j| - 1} = o(1)$ , that is  $\Re \beta_j \in (-1/2, 1/2)$  for all  $j = 0, 1, \dots, m$ . However, it is possible to obtain the solution in any half-closed or open interval of length 1, i.e. for  $|||\beta||| < 1$ , as follows.

Let  $|||\beta||| < 1$  and consider the transformation

$$(4.30) \quad \tilde{R}(z) = n^{\omega \sigma_3} R(z) n^{-\omega \sigma_3} \quad z \in \mathbb{C} \setminus \Gamma,$$

where

$$(4.31) \quad \omega = \frac{1}{2}(\min_j \Re \beta_j + \max_j \Re \beta_j)$$

which “shifts” all  $\Re \beta_j$  (in the conjugation  $n^{\beta_j}$  terms of (4.20) for the jump matrix in (4.29)) into the interval  $(-1/2, 1/2)$ . Note that  $\omega = \Re \beta_{j_0}$  if only one  $\Re \beta_{j_0} \neq 0$ , and  $\omega = 0$  if all  $\Re \beta_j = 0$ .

Now in the RHP for  $\tilde{R}(z)$ , the condition at infinity and the uniform exponential estimate  $I + O(\exp(-\varepsilon n))$  (with different  $\varepsilon$ ) of the jump matrices on  $\Sigma^{\text{out}}, \Sigma''^{\text{out}}$  is preserved, while the jump matrices on  $\partial U_{z_j}$  have the form:

$$(4.32) \quad I + n^{\omega \sigma_3} \Delta_1(z) n^{-\omega \sigma_3} + \dots + n^{\omega \sigma_3} \Delta_k(z) n^{-\omega \sigma_3} + n^{\omega \sigma_3} \Delta_{k+1}^{(r)}(z) n^{-\omega \sigma_3}, \quad z \in \partial U_{z_j},$$

where the order of each  $n^{\omega \sigma_3} \Delta_p(z) n^{-\omega \sigma_3}$ ,  $n^{\omega \sigma_3} \Delta_p^{(r)}(z) n^{-\omega \sigma_3}$ ,  $p = 1, 2, \dots$ ,  $z \in \cup_{j=0}^m \partial U_{z_j}$  is

$$O(n^{2 \max_j |\Re \beta_j - \omega| - p}) = O(n^{|||\beta||| - p}).$$

This implies that the standard analysis can be applied to the  $\tilde{R}$ -RHP problem in the range  $\Re \beta_j \in (q - 1/2, q + 1/2)$ ,  $j = 0, 1, \dots, m$ , for any  $q \in \mathbb{R}$ , and we obtain the asymptotic expansion

$$(4.33) \quad \tilde{R}(z) = I + \sum_{p=1}^k \tilde{R}_p(z) + \tilde{R}_{k+1}^{(r)}(z), \quad p = 1, 2, \dots$$

uniformly for all  $z$  and for  $\beta_j$  in bounded sets of the strip  $q - 1/2 < \Re \beta_j < q + 1/2$ ,  $j = 0, 1, \dots, m$ , i.e.  $|||\beta||| < 1$ , provided  $\alpha_j \pm \beta_j$  are outside neighborhoods of the points  $\alpha_j \pm \beta_j = -1, -2, \dots$  (cf. (4.21)).

The functions  $\tilde{R}_j(z)$  are computed recursively. We will need explicit expressions only for the first two. The first one is found from the conditions that  $\tilde{R}_1(z)$  is analytic outside  $\partial U = \cup_{j=0}^m \partial U_{z_j}$ ,  $\tilde{R}_1(z) \rightarrow 0$  as  $z \rightarrow \infty$ , and

$$(4.34) \quad \tilde{R}_{1,+}(z) = \tilde{R}_{1,-}(z) + n^{\omega \sigma_3} \Delta_1(z) n^{-\omega \sigma_3}, \quad z \in \partial U.$$

The solution is easily found. First denote

$$(4.35) \quad R_k(z) \equiv n^{-\omega \sigma_3} \tilde{R}_k(z) n^{\omega \sigma_3}, \quad R_k^{(r)}(z) \equiv n^{-\omega \sigma_3} \tilde{R}_k^{(r)}(z) n^{\omega \sigma_3}, \quad k = 1, 2, \dots,$$

and write for  $R$ :

$$(4.36) \quad R_1(z) = \frac{1}{2\pi i} \int_{\partial U} \frac{\Delta_1(x) dx}{x - z} \\ = \begin{cases} \sum_{k=0}^m \frac{A_k}{z - z_k}, & z \in \mathbb{C} \setminus \cup_{j=0}^m U_{z_j} \\ \sum_{k=0}^m \frac{A_k}{z - z_k} - \Delta_1(z), & z \in U_{z_j}, \quad j = 0, 1, \dots, m. \end{cases}, \quad \partial U = \cup_{j=0}^m \partial U_{z_j}.$$

where the contours in the integral are traversed in the negative direction, and  $A_k$  are the coefficients in the Laurent expansion of  $\Delta_1(z)$ :

$$(4.37) \quad \Delta_1(z) = \frac{A_k}{z - z_k} + B_k + O(z - z_k), \quad z \rightarrow z_k, \quad k = 0, 1, \dots, m.$$

The coefficients are easy to compute using (4.19) and (4.25):

$$(4.38) \quad A_k \equiv A_k^{(n)} = \frac{z_k}{n} \begin{pmatrix} -(\alpha_k^2 - \beta_k^2) & \frac{\Gamma(1+\alpha_k+\beta_k)}{\Gamma(\alpha_k-\beta_k)} z_k^n \mu_k^2 n^{-2\beta_k} \\ -\frac{\Gamma(1+\alpha_k-\beta_k)}{\Gamma(\alpha_k+\beta_k)} z_k^{-n} \mu_k^{-2} n^{2\beta_k} & \alpha_k^2 - \beta_k^2 \end{pmatrix}.$$

The function  $\tilde{R}_2$  is now found from the conditions that  $\tilde{R}_2(z) \rightarrow 0$  as  $z \rightarrow \infty$ , is analytic outside  $\partial U$ , and

$$(4.39) \quad \tilde{R}_{2,+}(z) = \tilde{R}_{2,-}(z) + \tilde{R}_{1,-}(z) n^{\omega\sigma_3} \Delta_1(z) n^{-\omega\sigma_3} + n^{\omega\sigma_3} \Delta_2(z) n^{-\omega\sigma_3}, \quad z \in \partial U.$$

The solution to this RHP is

$$(4.40) \quad \tilde{R}_2(z) = \frac{1}{2\pi i} \int_{\partial U} \left( \tilde{R}_{1,-}(x) n^{\omega\sigma_3} \Delta_1(x) n^{-\omega\sigma_3} + n^{\omega\sigma_3} \Delta_2(x) n^{-\omega\sigma_3} \right) \frac{dx}{x - z}.$$

At the  $k$ 'th step we have the RHP for  $\tilde{R}_k(z)$  with the same analyticity condition and the condition at infinity, and the following jump:

$$(4.41) \quad \tilde{R}_{k,+}(z) = \tilde{R}_{k,-}(z) + \sum_{p=1}^k \tilde{R}_{k-p,-}(z) n^{\omega\sigma_3} \Delta_p(z) n^{-\omega\sigma_3}, \quad z \in \partial U,$$

where  $\tilde{R}_0(z) \equiv I$ .

We will now discuss the way in which the general  $\tilde{R}_k(z)$  depends on  $n$ . In particular, we will discuss its order in  $n$ . First note that

$$(4.42) \quad \tilde{R}_1(z) \sim \frac{1}{n} \begin{pmatrix} 1 & \sum_j b_j^{-2} \\ \sum_j b_j^2 & 1 \end{pmatrix}, \quad \tilde{R}_2(z) \sim \frac{1}{n^2} \begin{pmatrix} 1 + \delta' n^2 & \sum_j b_j^{-2} \\ \sum_j b_j^2 & 1 + \delta' n^2 \end{pmatrix},$$

$$(4.43) \quad b_j \equiv n^{\beta_j - \omega} z_j^{-n/2}, \quad \delta' \sim \sum_{j,k} n^{2((\beta_j - \beta_k) - 1)} \left( \frac{z_k}{z_j} \right)^n.$$

Here the notation  $A \sim B$  means  $A' = B'$ , where  $X'$  is  $X$  in which each matrix element and each term in the sums is multiplied by a suitable constant *independent* of  $n$ . Starting with these expressions, and noting from (4.41) that

$$(4.44) \quad \tilde{R}_k(z) \sim \sum_{p=1}^k \tilde{R}_{k-p,-}(z) n^{\omega\sigma_3} \Delta_p(z) n^{-\omega\sigma_3} \sim \tilde{R}_{k-1,-}(z) n^{\omega\sigma_3} \Delta_1(z) n^{-\omega\sigma_3} + \frac{1}{n} \tilde{R}_{k-1,-}(z),$$

we obtain by induction:

$$(4.45) \quad \tilde{R}_{2p+1}(z) \sim \frac{1}{n^{2p+1}} \sum_{k=0}^p (\delta' n^2)^k \begin{pmatrix} 1 & \sum_j b_j^{-2} \\ \sum_j b_j^2 & 1 \end{pmatrix},$$

$$(4.46) \quad \tilde{R}_{2p+2}(z) \sim \frac{1}{n^{2p+2}} \sum_{k=0}^p (\delta' n^2)^k \begin{pmatrix} 1 + \delta' n^2 & \sum_j b_j^{-2} \\ \sum_j b_j^2 & 1 + \delta' n^2 \end{pmatrix}, \quad p = 0, 1, \dots$$

In particular,

$$(4.47) \quad \tilde{R}_{2p+1}(z) = \frac{\delta'^p}{n} O \begin{pmatrix} 1 & \sum_j b_j^{-2} \\ \sum_j b_j^2 & 1 \end{pmatrix}, \quad \tilde{R}_{2p+2}(z) = \frac{\delta'^p}{n^2} O \begin{pmatrix} 1 + \delta' n^2 & \sum_j b_j^{-2} \\ \sum_j b_j^2 & 1 + \delta' n^2 \end{pmatrix},$$

$$(4.48) \quad O(\delta') = O(\delta), \quad \delta = \max_{j,k} n^{2(\Re(\beta_j - \beta_k) - 1)} = n^{2(\|\beta\| - 1)}, \quad p = 0, 1, \dots,$$



as  $n \rightarrow \infty$ . Here  $O(A)$  represent  $2 \times 2$  matrices with elements of the corresponding order.

Finally, note that the error term in (4.33) is

$$(4.49) \quad \tilde{R}_k^{(r)}(z) = O(|\tilde{R}_k(z)| + |\tilde{R}_{k+1}(z)|).$$

In particular, as is clear from the above, if there is only one nonzero  $\beta_{j_0}$ , we obtain the expansion purely in inverse integer powers of  $n$  valid in fact for all  $\beta_{j_0} \in \mathbb{C}$  uniformly in bounded sets of the complex plane.

It is clear from the construction and the properties of the asymptotic series of the confluent hypergeometric function that the error terms  $\tilde{R}_k^{(r)}(z)$  are uniform for  $\beta_j$  in bounded subsets of the strip  $q - 1/2 < \Re \beta_j < q + 1/2$ ,  $j = 0, 1, \dots, m$ , for  $\alpha_j$  in bounded sets of the half-plane  $\Re \alpha_j > -1/2$ , and for  $\alpha_j \pm \beta_j$  away from neighborhoods of the negative integers. Moreover, the series (4.33) is differentiable in  $\alpha_j, \beta_j$ .

## 5. ASYMPTOTICS FOR DIFFERENTIAL IDENTITIES AND INTEGRATION. PROOF OF THEOREM 1.1

**5.1. Pure Fisher-Hartwig singularities. The case  $|||\beta||| < 1/2$ .** First, we will prove the theorem for  $V(z) \equiv 0$  and  $|||\beta||| = \max_{j,k} |\Re \beta_j - \Re \beta_k| < 1/2$ . The proof is based on the differential identity (3.20). First, we show that (3.20) has the following asymptotic form.

**Proposition 5.1.** *Let  $(\alpha_0, \beta_0, \dots, \alpha_m, \beta_m)$  be in a compact subset, denote it  $\Lambda$ , belonging to the subset  $|||\beta||| < 1$ ,  $\alpha_j \pm \beta_j \neq -1, -2, \dots$  of the parameter space  $\mathcal{P} = \{(\alpha_0, \beta_0, \dots, \alpha_m, \beta_m) : \alpha_j, \beta_j \in \mathbb{C}, \Re \alpha_j > -1/2\}$  and including the point  $\alpha_j = \beta_j = 0$ ,  $j = 0, 1, \dots, m$ . Let  $\beta_j = 0$  if  $\alpha_j = 0$ ,  $j = 0, 1, \dots, m$ ,  $\delta = n^{2(|||\beta|||-1)}$ . Then for  $n \rightarrow \infty$ , and  $\nu = 0, 1, \dots, m$ ,*

$$(5.1) \quad \frac{\partial}{\partial \alpha_\nu} \ln D_n(f(z)) = 2\alpha_\nu + (\alpha_\nu + \beta_\nu) \left[ \frac{\partial}{\partial \alpha_\nu} \ln \frac{\Gamma(1 + \alpha_\nu + \beta_\nu)}{\Gamma(1 + 2\alpha_\nu)} + \ln n \right] + (\alpha_\nu - \beta_\nu) \left[ \frac{\partial}{\partial \alpha_\nu} \ln \frac{\Gamma(1 + \alpha_\nu - \beta_\nu)}{\Gamma(1 + 2\alpha_\nu)} + \ln n \right] - \sum_{j \neq \nu} \left[ (\alpha_j + \beta_j) \ln \frac{z_j - z_\nu}{z_j} + (\alpha_j - \beta_j) \ln \frac{z_j - z_\nu}{z_\nu e^{i\pi}} \right] + 2\pi i \sum_{j=0}^{\nu-1} (\alpha_j + \beta_j) + O(n^{-1} \ln n) + O(\delta n \ln n)$$

and

$$(5.2) \quad \frac{\partial}{\partial \beta_\nu} \ln D_n(f(z)) = -2\beta_\nu + (\alpha_\nu + \beta_\nu) \left[ \frac{\partial}{\partial \beta_\nu} \ln \Gamma(1 + \alpha_\nu + \beta_\nu) - \ln n \right] + (\alpha_\nu - \beta_\nu) \left[ \frac{\partial}{\partial \beta_\nu} \ln \Gamma(1 + \alpha_\nu - \beta_\nu) + \ln n \right] + \sum_{j \neq \nu} \left[ (\alpha_j + \beta_j) \ln \frac{z_j - z_\nu}{z_j} - (\alpha_j - \beta_j) \ln \frac{z_j - z_\nu}{z_\nu e^{i\pi}} \right] - 2\pi i \sum_{j=0}^{\nu-1} (\alpha_j + \beta_j) + O(n^{-1} \ln n) + O(\delta n \ln n).$$

*Remark 5.2.* The error term  $O(\delta n \ln n) = o(1)$  uniformly in  $\Lambda$  if  $|||\beta||| \leq 1/2 - \varepsilon$ ,  $\varepsilon > 0$ . In fact, the estimate for the error term can be considerably improved: see next section.

*Remark 5.3.* The case  $\beta_j = 0$  if  $\alpha_j = 0$  is all we need below. After the proof of Proposition 5.1, we will integrate the identity (5.1) to obtain the asymptotics of  $D_n$  for all  $\alpha_j \neq 0$ ,  $\beta_j = 0$ . We then integrate the identity (5.2) and obtain the asymptotics of  $D_n$  for all  $\alpha_j \neq 0$ ,  $\beta_j \neq 0$ . This gives the general result for  $V \equiv 0$ ,  $|||\beta||| < 1/2$ , since we can set any  $\alpha_j = 0$  using the uniformity of the asymptotic expansion in the  $\alpha_j$ 's.

*Proof.* Assume that for all  $j$ ,  $\beta_j = 0$  if  $\alpha_j = 0$ , and  $D_k(f) \neq 0$ ,  $k = 1, 2, \dots$ . Then we can rewrite (3.20) in the form:

$$(5.3) \quad \frac{\partial}{\partial \gamma} \ln D_n(f(z)) = -2 \frac{\frac{\partial \chi_n}{\partial \gamma}}{\chi_n} \left( n + \sum_{j=0}^m \alpha_j \left\{ 1 - \tilde{Y}_{12}^{(n)}(z_j) Y_{21}^{(n+1)}(z_j) - z_j Y_{11}^{(n)}(z_j) \tilde{Y}_{22}^{(n+1)}(z_j) \right\} \right) \\ - 2 \sum_{j=0}^m \alpha_j \left( \tilde{Y}_{12}^{(n)}(z_j) \frac{\partial}{\partial \gamma} Y_{21}^{(n+1)}(z_j) - z_j \frac{\partial}{\partial \gamma} Y_{11}^{(n)}(z_j) \tilde{Y}_{22}^{(n+1)}(z_j) \right),$$

where  $\gamma = \alpha_j$  or  $\gamma = \beta_j$ . We now estimate the right-hand side of this identity as  $n \rightarrow \infty$ . The asymptotics of  $\chi_n$  were found in ([15], Theorem 1.8). We need these asymptotics here in the case  $V \equiv 0$ :

$$(5.4) \quad \chi_{n-1}^2 = 1 - \frac{1}{n} \sum_{k=0}^m (\alpha_k^2 - \beta_k^2) \\ + \sum_{j=0}^m \sum_{k \neq j} \frac{z_k}{z_j - z_k} \left( \frac{z_j}{z_k} \right)^n n^{2(\beta_k - \beta_j - 1)} \frac{\nu_j}{\nu_k} \frac{\Gamma(1 + \alpha_j + \beta_j) \Gamma(1 + \alpha_k - \beta_k)}{\Gamma(\alpha_j - \beta_j) \Gamma(\alpha_k + \beta_k)} \\ + O(\delta^2) + O(\delta/n), \quad \delta = n^{2(\|\beta\| - 1)}, \quad n \rightarrow \infty,$$

where

$$(5.5) \quad \nu_j = \exp \left\{ -i\pi \left( \sum_{p=0}^{j-1} \alpha_p - \sum_{p=j+1}^m \alpha_p \right) \right\} \prod_{p \neq j} \left( \frac{z_j}{z_p} \right)^{\alpha_p} |z_j - z_p|^{2\beta_p}.$$

The asymptotics of  $\tilde{Y}^{(n)}(z_j)$  were also found in [15] (Eq. (7.11)-(7.21)). Namely,

$$(5.6) \quad \tilde{Y}^{(n)}(z_j) = (I + r_j^{(n)}) L_j^{(n)}, \quad L_j^{(n)} = \begin{pmatrix} M_{21} \mu_j \eta_j^{-1} n^{\alpha_j - \beta_j} z_j^n & M_{22} \mu_j \eta_j n^{-\alpha_j - \beta_j} \\ -M_{11} \mu_j^{-1} \eta_j^{-1} n^{\alpha_j + \beta_j} & -M_{12} \mu_j^{-1} \eta_j n^{-\alpha_j + \beta_j} z_j^{-n} \end{pmatrix},$$

where  $r_j = R_1^{(r)}(z_j)$ , the parameters  $\eta_j, \mu_j$  are given by (4.23), (4.25), and

$$M = \begin{pmatrix} \frac{\Gamma(1 + \alpha_j - \beta_j)}{\Gamma(1 + 2\alpha_j)} & -\frac{\Gamma(2\alpha_j)}{\Gamma(\alpha_j + \beta_j)} \\ \frac{\Gamma(1 + \alpha_j + \beta_j)}{\Gamma(1 + 2\alpha_j)} & \frac{\Gamma(2\alpha_j)}{\Gamma(\alpha_j - \beta_j)} \end{pmatrix}.$$

Note that the matrix  $L_j^{(n)}$  has the structure

$$(5.7) \quad L_j^{(n)} = n^{-\beta_j \sigma_3} \hat{L}_j^{(n)} n^{\alpha_j \sigma_3},$$

where  $\hat{L}$  depends on  $n$  only via the oscillatory terms  $z_j^n$ .

Let us now obtain the asymptotics for the following combination appearing in (5.3):

$$(5.8) \quad \frac{\partial}{\partial \gamma} Y_{11}^{(n)}(z_j) \tilde{Y}_{22}^{(n+1)}(z_j) = \frac{\partial}{\partial \gamma} \left( (1 + r_{11}^{(n)}) L_{11}^{(n)} + r_{12}^{(n)} L_{21}^{(n)} \right) \left( r_{21}^{(n+1)} L_{12}^{(n+1)} + (1 + r_{22}^{(n+1)}) L_{22}^{(n+1)} \right) \\ = \left( L_{11}^{(n)} L_{22}^{(n+1)} (1 + r_{22}^{(n+1)}) + L_{11}^{(n)} L_{12}^{(n+1)} r_{21}^{(n+1)} \right) \left[ (1 + r_{11}^{(n)}) \frac{\partial}{\partial \gamma} \ln L_{11}^{(n)} + \frac{\partial}{\partial \gamma} r_{11}^{(n)} \right] \\ + \left( L_{21}^{(n)} L_{22}^{(n+1)} (1 + r_{22}^{(n+1)}) + L_{21}^{(n)} L_{12}^{(n+1)} r_{21}^{(n+1)} \right) \left[ r_{12}^{(n)} \frac{\partial}{\partial \gamma} \ln L_{21}^{(n)} + \frac{\partial}{\partial \gamma} r_{12}^{(n)} \right].$$

We omit the lower index  $j$  of  $r$  and  $L$  for simplicity of notation.

Now the explicit formula for  $L$  and the estimates for  $\tilde{R}(z)$  imply that

$$(5.9) \quad L_{11}^{(n)} L_{22}^{(n+1)} = O(1), \quad L_{21}^{(n)} L_{12}^{(n+1)} = O(1),$$

$$(5.10) \quad L_{11}^{(n)} L_{12}^{(n+1)} r_{21}^{(n+1)} = O(n\delta), \quad L_{21}^{(n)} L_{22}^{(n+1)} r_{12}^{(n)} = O(n\delta), \quad r_{21}^{(n+1)} r_{12}^{(n)} = O(\delta),$$

$$(5.11) \quad \frac{\partial}{\partial \gamma} \ln L^{(n)} = O(\ln n), \quad \frac{\partial}{\partial \gamma} r = O(r) \ln n,$$

where as before

$$\delta = n^{2(\|\beta\| - 1)}.$$

Therefore,

$$(5.12) \quad \begin{aligned} \frac{\partial}{\partial \gamma} Y_{11}^{(n)}(z_j) \tilde{Y}_{22}^{(n+1)}(z_j) &= L_{11}^{(n)} L_{22}^{(n+1)} \frac{\partial}{\partial \gamma} \ln L_{11}^{(n)} + O\left(\frac{\ln n}{n}\right) + O(\delta n \ln n) \\ &= \frac{\alpha_j + \beta_j}{2\alpha_j z_j} \frac{\partial}{\partial \gamma} \ln L_{11}^{(n)} + O\left(\frac{\ln n}{n}\right) + O(\delta n \ln n). \end{aligned}$$

Similarly, we obtain

$$(5.13) \quad \tilde{Y}_{12}^{(n)}(z_j) \frac{\partial}{\partial \gamma} Y_{21}^{(n+1)}(z_j) = -\frac{\alpha_j - \beta_j}{2\alpha_j} \frac{\partial}{\partial \gamma} \ln L_{21}^{(n+1)} + O\left(\frac{\ln n}{n}\right) + O(\delta n \ln n),$$

and furthermore,

$$(5.14) \quad \tilde{Y}_{12}^{(n)}(z_j) Y_{21}^{(n+1)}(z_j) = O(\delta n) + O(1), \quad Y_{11}^{(n)}(z_j) \tilde{Y}_{22}^{(n+1)}(z_j) = O(\delta n) + O(1).$$

Note that because of the special structure of (5.7), the quantity  $n^{\alpha_j}$  does not appear in any of the products (5.12)–(5.14). Substituting (5.12)–(5.14) into (5.3) and using the asymptotics (5.4), we obtain

$$(5.15) \quad \begin{aligned} \frac{\partial}{\partial \gamma} \ln D_n(f(z)) &= \frac{\partial}{\partial \gamma} \left[ \sum_{j=0}^m (\alpha_j^2 - \beta_j^2) \right] \\ &+ \sum_{j=0}^m \left[ (\alpha_j + \beta_j) \frac{\partial}{\partial \gamma} \ln L_{j,11}^{(n)} + (\alpha_j - \beta_j) \frac{\partial}{\partial \gamma} \ln L_{j,21}^{(n+1)} \right] + O\left(\frac{\ln n}{n}\right) + O(\delta n \ln n). \end{aligned}$$

Let us calculate the logarithmic derivatives appearing in (5.15). From (5.6), (4.25), and (4.23) it is easy to obtain for the derivatives w.r.t.  $\alpha_\nu$ ,  $\nu = 0, 1, \dots, m$ :

$$(5.16) \quad \frac{\partial}{\partial \alpha_\nu} \ln L_{\nu,11}^{(n)} = \frac{\partial}{\partial \alpha_\nu} \ln \frac{\Gamma(1+\alpha_\nu+\beta_\nu)}{\Gamma(1+2\alpha_j)} + \ln n;$$

$$(5.17) \quad \frac{\partial}{\partial \alpha_\nu} \ln L_{j,11}^{(n)} = -\ln \frac{z_j - z_\nu}{z_j} + 2\pi i, \quad j < \nu; \quad \frac{\partial}{\partial \alpha_\nu} \ln L_{j,11}^{(n)} = -\ln \frac{z_j - z_\nu}{z_j}, \quad j > \nu;$$

$$(5.18) \quad \frac{\partial}{\partial \alpha_\nu} \ln L_{\nu,21}^{(n+1)} = \frac{\partial}{\partial \alpha_\nu} \ln \frac{\Gamma(1+\alpha_\nu-\beta_\nu)}{\Gamma(1+2\alpha_j)} + \ln n;$$

$$(5.19) \quad \frac{\partial}{\partial \alpha_\nu} \ln L_{j,21}^{(n+1)} = -\ln \frac{z_j - z_\nu}{z_\nu e^{i\pi}}, \quad j \neq \nu.$$

Similarly, we obtain for the derivatives w.r.t.  $\beta_\nu$ :

$$(5.20) \quad \frac{\partial}{\partial \beta_\nu} \ln L_{\nu,11}^{(n)} = \frac{\partial}{\partial \beta_\nu} \ln \Gamma(1 + \alpha_\nu + \beta_\nu) - \ln n;$$

$$(5.21) \quad \frac{\partial}{\partial \beta_\nu} \ln L_{j,11}^{(n)} = \ln \frac{z_j - z_\nu}{z_j} - 2\pi i, \quad j < \nu; \quad \frac{\partial}{\partial \beta_\nu} \ln L_{j,11}^{(n)} = \ln \frac{z_j - z_\nu}{z_j}, \quad j > \nu;$$

$$(5.22) \quad \frac{\partial}{\partial \beta_\nu} \ln L_{\nu,21}^{(n+1)} = \frac{\partial}{\partial \beta_\nu} \ln \Gamma(1 + \alpha_\nu - \beta_\nu) + \ln n;$$

$$(5.23) \quad \frac{\partial}{\partial \beta_\nu} \ln L_{j,21}^{(n+1)} = -\ln \frac{z_j - z_\nu}{z_\nu e^{i\pi}}, \quad j \neq \nu.$$

Combining these results with (5.15) we obtain (5.1), (5.2) on condition that  $D_k(f) \neq 0$ ,  $k = 1, 2, \dots$ , or equivalently (see Section 2),  $(\alpha_0, \beta_0, \dots, \alpha_m, \beta_m) \in \Lambda \setminus \Omega_{k_0}$ . This condition can be replaced simply by  $(\alpha_0, \beta_0, \dots, \alpha_m, \beta_m) \in \Lambda$  in the following way. Let

$$(5.24) \quad \beta_0 = \alpha_1 = \dots = \alpha_m = \beta_m = 0.$$

Let  $\Omega_{k_0}(\alpha_0)$  be the subset of  $\Omega_{k_0}$  with  $\alpha_j, \beta_j$  fixed by (5.24). Since  $D_k(f) \equiv D_k(f(\alpha_0; z))$  is an analytic function of  $\alpha_0$  and  $D_k(1) \neq 0$ , the set  $\Omega_{k_0}(\alpha_0)$  is finite. Let us rewrite the identity (5.1) with  $\nu = 0$  and assuming (5.24) in the form  $H'(\alpha_0) = 0$ , where  $H(\alpha_0) = D_n(f(\alpha_0; z)) \exp(-\int_0^{\alpha_0} r(n, s) ds)$  and where  $r(n, \alpha_0)$  is the r.h.s. of (3.20) with  $\gamma = \alpha_0$  and assuming (5.24). Since the expression (5.1) for  $r(n, \alpha_0)$  holds uniformly and is continuous for  $\alpha_0 \in \Lambda$  provided  $n$  is larger than some  $k_0(\Lambda)$ , and  $D_n(f(\alpha_0; z))$  and its derivative are continuous, the function  $H(\alpha_0)$  is continuously differentiable for all  $n > k_0(\Lambda)$ . Hence,  $H'(\alpha_0) = 0$  for all  $\alpha_0 \in \Lambda$  and  $n > k_0(\Lambda)$ . Taking into account that  $H(0) = D_n(1) \neq 0$ , we conclude that  $D_n(f(\alpha_0; z))$  is nonzero, and that the identity (5.1) under (5.24) is, in fact, true for all  $\alpha_0 \in \Lambda$  if  $n$  is sufficiently large (larger than  $k_0(\Lambda)$ ). Now fix  $\alpha_0 \in \Lambda$  and assume the condition  $\alpha_1 = \dots = \alpha_m = \beta_m = 0$ . A similar argument as above then gives that  $D_n(f(\alpha_0, \beta_0; z))$  is nonzero and the identity (5.2) with  $\nu = 0$  is true for all  $\beta_0 \in \Lambda$  if  $n$  is sufficiently large. Continuing this way, we complete the proof of Proposition 5.1 by induction.

*Remark 5.4.* A similar argument applies to the asymptotic form of the differential identity (3.26) we need in Section 5.3 below. We omit the discussion. □

We will now complete the proof of Theorem 1.1 in the case  $|||\beta||| < 1/2$ ,  $V(z) \equiv 0$  by integrating the identities of Proposition 5.1. In this case we denote

$$D_n(f(z)) = D_n(\alpha_0, \dots, \alpha_m; \beta_0, \dots, \beta_m).$$

First, set  $m = 0$  and  $\beta_0 = 0$ . Then (5.1) becomes

$$(5.25) \quad \frac{\partial}{\partial \alpha_0} \ln D_n(\alpha_0) = 2\alpha_0 \left( 1 + \ln n + \frac{d}{d\alpha_0} \frac{\Gamma(1 + \alpha_0)}{\Gamma(1 + 2\alpha_0)} \right) + O(n^{-1} \ln n) + O(\delta n \ln n).$$

Integrating both sides over  $\alpha_0$  from 0 to some  $\alpha_0$  and using the fact that  $D_n(0) = 1$ , we obtain

$$(5.26) \quad D_n(\alpha_0) = n^{\alpha_0^2} \frac{G(1 + \alpha_0)^2}{G(1 + 2\alpha_0)},$$

where  $G(x)$  is Barnes  $G$ -function. To perform the integration we used the identity

$$(5.27) \quad \int_0^z \left( 1 + \frac{d}{dx} \frac{\Gamma(1 + x)}{\Gamma(1 + 2x)} \right) 2x dx = \ln \frac{G(1 + z)^2}{G(1 + 2z)},$$

which easily follows from the standard formula (see, e.g. [28]):

$$(5.28) \quad \int_0^z \ln \Gamma(x + 1) dx = \frac{z}{2} \ln 2\pi - \frac{z(z + 1)}{2} + z \ln \Gamma(z + 1) - \ln G(z + 1).$$

Now set  $m = 1$ ,  $\alpha_0$  fixed. Set  $\beta_0 = \beta_1 = 0$ . Relation (5.1) for  $\nu = 1$  is then

$$(5.29) \quad \frac{\partial}{\partial \alpha_1} \ln D_n(\alpha_0, \alpha_1) = 2\alpha_1 \left( 1 + \ln n + \frac{d}{d\alpha_0} \frac{\Gamma(1 + \alpha_1)}{\Gamma(1 + 2\alpha_1)} \right) + \alpha_0 \ln z_0 + \alpha_0 (\ln z_1 + i\pi) - 2\alpha_0 \ln(z_0 - z_1) + 2\pi i \alpha_0 + O(n^{-1} \ln n) + O(\delta n \ln n).$$

Integrating this over  $\alpha_1$  from 0 to some fixed  $\alpha_1$  along a path lying in  $\Lambda$  (see Proposition 5.1) and using (5.27), we obtain

$$(5.30) \quad \ln \frac{D_n(\alpha_0, \alpha_1)}{D_n(\alpha_0, 0)} = \alpha_1^2 \ln n + 2 \ln G(1 + \alpha_1) - \ln G(1 + 2\alpha_1) + \alpha_0 \alpha_1 \ln(z_0 z_1 e^{3\pi i}) \\ - 2\alpha_0 \alpha_1 \ln(z_0 - z_1) + O(n^{-1} \ln n) + O(\delta n \ln n).$$

Substituting here (5.26), we obtain

$$(5.31) \quad D_n(\alpha_0, \alpha_1) = n^{\alpha_0^2 + \alpha_1^2} \prod_{j=0}^1 \frac{G(1 + \alpha_j)^2}{G(1 + 2\alpha_j)} \left[ \frac{(z_0 - z_1)^2}{z_0 z_1 e^{3\pi i}} \right]^{-\alpha_0 \alpha_1} (1 + O(n^{-1} \ln n) + O(\delta n \ln n)) = \\ n^{\alpha_0^2 + \alpha_1^2} \prod_{j=0}^1 \frac{G(1 + \alpha_j)^2}{G(1 + 2\alpha_j)} |z_0 - z_1|^{-\alpha_0 \alpha_1} (1 + O(n^{-1} \ln n) + O(\delta n \ln n))$$

(to write the last equation we recall (3.13), the way the branch of  $(z - z_j)^{\alpha_j}$  was fixed there, and the fact that  $\arg z_0 < \arg z_1$ ).

Continuing this way, we finally obtain by induction for any fixed  $m$ ,  $\beta_j = 0$ ,  $j = 0, 1, \dots, m$ , the asymptotic expression

$$(5.32) \quad D_n(\alpha_0, \dots, \alpha_m) = n^{\sum_{j=0}^m \alpha_j^2} \prod_{j=0}^m \frac{G(1 + \alpha_j)^2}{G(1 + 2\alpha_j)} \prod_{0 \leq j < k \leq m} |z_j - z_k|^{-\alpha_j \alpha_k} (1 + O(n^{-1} \ln n) + O(\delta n \ln n)).$$

We now add in the  $\beta$ -singularities. We will make use of one more identity, which follows from (5.28):

$$(5.33) \quad \int_0^\beta \left( (\alpha + x) \frac{d}{dx} \ln \Gamma(1 + \alpha + x) + (\alpha - x) \frac{d}{dx} \ln \Gamma(1 + \alpha - x) - 2x \right) dx \\ = \ln \frac{G(1 + \alpha + \beta) G(1 + \alpha - \beta)}{G(1 + \alpha)^2}.$$

First, we obtain the result for the case when  $-1/4 < \Re \beta_j < 1/4$ . This implies that the order of the error term  $O(\delta n \ln n)$  remains  $o(1)$  as we integrate over  $\beta_j$ 's starting at zero. As before, we always assume integration along a path in  $\Lambda$ . Setting  $\nu = 0$ ,  $\beta_j = 0$ ,  $j = 1, \dots, m$  in (5.2), and integrating this identity over  $\beta_0$  from zero to a fixed  $\beta_0$ , we obtain using (5.33):

$$(5.34) \quad \ln \frac{D_n(\alpha_0, \dots, \alpha_m; \beta_0)}{D_n(\alpha_0, \dots, \alpha_m; 0)} = -\beta_0^2 \ln n + \ln \frac{G(1 + \alpha_0 + \beta_0) G(1 + \alpha_0 - \beta_0)}{G(1 + \alpha_0)^2} \\ + \beta_0 \sum_{j \neq 0} \alpha_j \ln \frac{z_0 e^{i\pi}}{z_j} + O(n^{-1} \ln n) + O(\delta n \ln n).$$

Substituting here (5.32), we obtain

$$(5.35) \quad D_n(\alpha_0, \dots, \alpha_m; \beta_0) = n^{\sum_{j=0}^m \alpha_j^2 - \beta_0^2} \frac{G(1 + \alpha_0 + \beta_0) G(1 + \alpha_0 - \beta_0)}{G(1 + \alpha_0)^2} \prod_{j=1}^m \frac{G(1 + \alpha_j)^2}{G(1 + 2\alpha_j)} \\ \times \prod_{0 \leq j < k \leq m} |z_j - z_k|^{-\alpha_j \alpha_k} \prod_{j=1}^m \left( \frac{z_0 e^{i\pi}}{z_j} \right)^{\alpha_j \beta_0} (1 + O(n^{-1} \ln n) + O(\delta n \ln n)).$$

Next, set  $\nu = 1$ ,  $\beta_j = 0$ ,  $j = 2, \dots, n$  in (5.2) and integrate over  $\beta_1$ . We obtain then the determinant  $D_n(\alpha_0, \dots, \alpha_m; \beta_0, \beta_1)$ . Continuing this procedure, we finally obtain by induction at step  $m$  the

asymptotics (1.10) for the case  $-1/4 < \Re\beta_j < 1/4$  with  $V \equiv 0$  and the error term  $O(n^{-1} \ln n) + O(\delta n \ln n) = o(1)$ .

Consider now the general case  $|||\beta||| < 1/2$ . We can choose  $q$  such that all  $\Re\beta_j \in (q-1/4, q+1/4)$ . Divide  $(0, q)$  into subintervals of length less than  $1/2$ . Apply the above integration procedure to move all  $\beta_j$  from zero to the line where the real part of all  $\beta_j$  is the right end of the first subinterval. Since the length of subintervals is less than  $1/2$ , the error term  $O(\delta \ln n)$  remains  $o(1)$ . Recall that during the integration we avoid any points where  $\alpha_j + \beta_j$  or  $\alpha_j - \beta_j$  is a negative integer. Next, move the  $\beta_j$ 's to the right end of the second subinterval, and so on, until the point  $\Re\beta_j = q$ . From that point move the  $\beta_j$ 's as needed. We thus obtain Theorem 1.1 with  $V \equiv 0$  and  $|||\beta||| < 1/2$ .

**5.2. Pure Fisher-Hartwig singularities. Extension to  $|||\beta||| < 1$ .** We now show that in fact the error term in (1.10) remains  $o(1)$  for the full range  $|||\beta||| < 1$ . First, recall the definition of  $\tilde{R}$  and  $\omega$  in (4.30) and (4.31). We have  $|||\beta||| = 2 \max_j (\Re\beta_j - \omega)$  and

$$-\frac{1}{2} < \Re\beta_j - \omega < \frac{1}{2}$$

for all singular points  $z_j$ . We denote  $p_j = z_j$  if  $\Re\beta_j - \omega > 0$ , and  $m_+$  the number of such points. Furthermore, denote  $q_j = z_j$  if  $\Re\beta_j - \omega < 0$ , and  $m_-$  the number of such points. Finally, let  $r_j = z_j$  if  $\Re\beta_j - \omega = 0$ .

Separating the main contributions in  $n$  (see (4.21)), we write the jump matrix for  $\tilde{R}$  on  $\partial U_{z_j}$  (cf. (4.32)) in the form

$$(5.36) \quad I + n^{\omega\sigma_3} \Delta_1(z) n^{-\omega\sigma_3} + \dots = I + \hat{\Delta}_1(z) + \hat{D}(z) + O(n^{-1-\rho}), \quad z \in \partial U_{z_j},$$

where

$$(5.37) \quad \rho = 1 - |||\beta|||$$

and

$$(5.38) \quad \hat{D}(z) = n^{\omega\sigma_3} \Delta_1(z) n^{-\omega\sigma_3} - \hat{\Delta}_1, \quad z \in \partial U_{z_j},$$

$$(5.39) \quad \hat{\Delta}_1(z) = (n^{\omega\sigma_3} \Delta_1(z) n^{-\omega\sigma_3})_{21} \sigma_- = \frac{b_j(z)}{z - p_j} \sigma_-, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$b_j(z) = -\frac{\Gamma(1 + \alpha_j - \beta_j)}{\Gamma(\alpha_j + \beta_j)} \left( \frac{\mathcal{D}(z)}{\zeta^{\beta_j} F_j(z)} \right)^{-2} p_j^{-n} e^{-i\pi(2\beta_j - \alpha_j)} \frac{z - p_j}{n \ln(z/p_j)} n^{-2\omega}, \quad z \in \partial U_{p_j},$$

$$(5.40) \quad \hat{\Delta}_1(z) = (n^{\omega\sigma_3} \Delta_1(z) n^{-\omega\sigma_3})_{12} \sigma_+ = \frac{a_j(z)}{z - q_j} \sigma_+, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$a_j(z) = \frac{\Gamma(1 + \alpha_j + \beta_j)}{\Gamma(\alpha_j - \beta_j)} \left( \frac{\mathcal{D}(z)}{\zeta^{\beta_j} F_j(z)} \right)^2 q_j^n e^{i\pi(2\beta_j - \alpha_j)} \frac{z - q_j}{n \ln(z/q_j)} n^{2\omega}, \quad z \in \partial U_{q_j}.$$

$$(5.41) \quad \hat{\Delta}_1(z) = 0, \quad z \in \partial U_{r_j}.$$

Note that  $\hat{D}(z)$  and  $\hat{\Delta}_1(z)$  are meromorphic functions in a neighborhood of  $U_{z_j}$  with a simple pole at  $z = z_j$ . We have (see (4.24), (4.25), and recall that  $V \equiv 0$ )

$$(5.42) \quad b_j = b_j^{(n)} \equiv \lim_{z \rightarrow p_j} b_j(z) = -n^{2(\beta_j - \omega) - 1} p_j^{-n+1} \mu_j^{-2} \frac{\Gamma(1 + \alpha_j - \beta_j)}{\Gamma(\alpha_j + \beta_j)} = O(n^{-\rho}),$$

$$(5.43) \quad a_j = a_j^{(n)} \equiv \lim_{z \rightarrow q_j} a_j(z) = n^{-2(\beta_j - \omega) - 1} q_j^{n+1} \mu_j^2 \frac{\Gamma(1 + \alpha_j + \beta_j)}{\Gamma(\alpha_j - \beta_j)} = O(n^{-\rho}).$$

Also note that

$$(5.44) \quad \widehat{D}(z) = O(n^{-1}), \quad z \in \partial U_{z_j}.$$

The main idea which will allow us to give the required estimate for the error term in (1.10) is the following. Write  $\widetilde{R}$  in the form

$$\widetilde{R}(z) = Q(z)\widehat{R}(z),$$

where  $\widehat{R}(z)$  is the solution to the RHP:

$$(5.45) \quad \widehat{R}(z) \text{ is analytic for } z \in \mathbb{C} \setminus \cup_j \partial U_{z_j}$$

$$(5.46) \quad \widehat{R}(z)_+ = \widehat{R}(z)_-(I + \widehat{\Delta}_1), \quad z \in \cup_j \partial U_{z_j},$$

$$(5.47) \quad \widehat{R}(z) = I + O(1/z), \quad z \rightarrow \infty.$$

We solve this RHP below explicitly, however, note first that the solution exists and is unique and

$$(5.48) \quad \widehat{R}(z) = I + O(n^{-\rho})$$

uniformly in  $z$  by standard arguments. We then have on  $\partial U_{z_j}$  using (5.48), the jump condition (5.46), and the nilpotency of  $\widehat{\Delta}_1$ :

$$Q_+ = \widetilde{R}_+ \widehat{R}_+^{-1} = \widetilde{R}_-(I + \widehat{\Delta}_1(z) + \widehat{D}(z) + O(n^{-1-\rho}))\widehat{R}_+^{-1} = Q_-(I + \widehat{D}(z) + O(n^{-1-\rho})).$$

The jump matrix for  $Q$  on  $\Sigma^{\text{out}}, \Sigma''^{\text{out}}$  remains exponentially close to the identity. Therefore,

$$(5.49) \quad Q(z) = I + Q_1(z) + O(n^{-1-\rho}), \quad Q_1(z) = \frac{1}{2\pi i} \int_{\cup_j \partial U_{z_j}} \frac{\widehat{D}(s)}{s-z} ds$$

In what follows, we will be interested in the matrix element  $Y_{21}(z)$  of our original RHP in a neighborhood of  $z = 0$ . In this neighborhood, we have using (5.49),

$$(5.50) \quad Y(z) = R(z)N(z) = n^{-\omega\sigma_3}Q(z)\widehat{R}(z)n^{\omega\sigma_3}N(z) = n^{-\omega\sigma_3}(I + Q_1(z) + O(n^{-1-\rho}))\widehat{R}(z)n^{\omega\sigma_3}\mathcal{D}(z)^{\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

From here, noting that  $\mathcal{D}(0) = 1$  as  $V(z) \equiv 0$ , and using the estimates  $\widehat{R}(z) = I + O(n^{-\rho})$ ,  $Q_1(0) = O(n^{-1})$ , we obtain

$$\chi_{n-1}^2 = -Y_{21}(0) = (1 + Q_{1,22}(0) + O(n^{-1-\rho}))\widehat{R}_{22}(0).$$

Using the expression for  $Q_1$  from (5.49) and (5.38), we can write this equation in the form (recall that the contours  $\partial U_{z_j}$  are oriented in the negative direction):

$$(5.51) \quad \chi_{n-1}^2 = \left( 1 - \frac{1}{n} \sum_{j=0}^m (\alpha_j^2 - \beta_j^2) + O(n^{-1-\rho}) \right) \widehat{R}_{22}(0).$$

Eventually, we will use the product of these quantities over  $n$  to represent the determinant. Before doing that, we now solve the RHP for  $\widehat{R}$  and find  $\widehat{R}_{22}(0)$ .

Set

$$(5.52) \quad \Phi(z) = \begin{cases} \widehat{R}(z) & z \in \mathbb{C} \setminus \cup_j U_{z_j} \\ \widehat{R}(z)(I + \widehat{\Delta}_1(z)) & z \in \cup_j U_{z_j}. \end{cases}$$

So defined,  $\Phi(z)$  is obviously a meromorphic function with simple poles at the points  $z_j$ , which tends to  $I$  at infinity. Therefore, it can be written in the form

$$(5.53) \quad \Phi(z) = I + \sum_{j=1}^{m_+} \frac{\Phi_j^+}{z - p_j} + \sum_{j=1}^{m_-} \frac{\Phi_j^-}{z - q_j}$$

for some constant matrices  $\Phi_j^\pm$ .

Moreover, the function  $\widehat{R}(z)$  (we use here the definition of  $\widehat{\Delta}_1$ ) in  $U_{p_k}$ ,  $k = 1, \dots, m_+$ ,

$$(5.54) \quad \widehat{R}(z) = \Phi(z)(I + \widehat{\Delta}_1)^{-1} = \left[ I + \sum_{j=1}^{m_+} \frac{\Phi_j^+}{z - p_j} + \sum_{j=1}^{m_-} \frac{\Phi_j^-}{z - q_j} \right] \left( I - \frac{b_k(z)}{z - p_k} \sigma_- \right)$$

is analytic in  $U_{p_k}$ . Hence, the coefficients at negative powers of  $z - p_k$  vanish. Equating the coefficient at  $(z - p_k)^{-2}$  to zero, we obtain

$$(5.55) \quad \Phi_k^+ \sigma_- = 0,$$

and therefore the matrix  $\Phi_k^+$  has the form

$$(5.56) \quad \Phi_k^+ = \begin{pmatrix} g_k & 0 \\ f_k & 0 \end{pmatrix}$$

for some constants  $g_k, f_k$ . The vanishing of the coefficient at  $(z - p_k)^{-1}$  gives the following condition on  $\Phi_j^\pm$ , where we used the equation  $\Phi_j^+ \sigma_- = 0$ ,

$$(5.57) \quad \Phi_k^+ - b_k \sum_{j=1}^{m_-} \frac{\Phi_j^- \sigma_-}{p_k - q_j} = b_k \sigma_-, \quad k = 1, \dots, m_+,$$

with  $b_k$  given by (5.42). Similarly, using the analyticity of  $\Phi(z)(I + \widehat{\Delta}_1(z))^{-1}$  in  $U_{q_k}$ , we obtain for all  $k = 1, \dots, m_-$

$$(5.58) \quad \Phi_k^- = \begin{pmatrix} 0 & e_k \\ 0 & h_k \end{pmatrix},$$

$$(5.59) \quad \Phi_k^- - a_k \sum_{j=1}^{m_+} \frac{\Phi_j^+ \sigma_+}{q_k - p_j} = a_k \sigma_+$$

for some constants  $e_k, h_k$ . Conditions (5.59), (5.57) are equations for the constants  $g_k, f_k, e_k, h_k$ . In view of (5.51), we are interested in

$$(5.60) \quad \widehat{R}_{22}(0) = \Phi_{22}(0) = 1 - \sum_{j=1}^{m_-} \frac{h_j}{q_j},$$

where we used (5.53) to write the last equation. To calculate this quantity we first substitute  $\Phi_j^+$  from (5.57) into (5.59). Then the 22 element of the resulting equations for  $k = 1, \dots, m_-$  can be written as follows:

$$(5.61) \quad (I - A)h = B,$$

where  $h$  and  $B$  are  $m_-$ -dimensional vectors with components  $h_k$  and

$$B_k = \sum_{j=1}^{m_+} \frac{a_k b_j}{q_k - p_j}, \quad k = 1, \dots, m_-,$$



$A = A^{(n)}$  is an  $m_- \times m_-$  matrix with matrix elements

$$A_{k,\ell} = \sum_{j=1}^{m_+} \frac{a_k b_j}{(q_k - p_j)(p_j - q_\ell)},$$

and  $I$  is the  $m_- \times m_-$  identity matrix.

Define the  $m_- \times m_-$  diagonal matrix  $\Delta$  as follows

$$\Delta = \text{diag}\{-q_1, -q_2, \dots, -q_{m_-}\}.$$

Then (5.61) can be written in the form

$$Tx = y, \quad T = I - \Delta^{-1}A\Delta, \quad x = \Delta^{-1}h, \quad y = \Delta^{-1}B$$

By Cramer's rule

$$x_k = \frac{\det(T_1 \cdots y \cdots T_{m_-})}{\det T},$$

where  $T_j$  are the columns of  $T$  and  $y$  is in the place of the  $k$ 'th column. We are interested in

$$(5.62) \quad 1 - \sum_{j=1}^{m_-} \frac{h_j}{q_j} = 1 + \sum_{j=1}^{m_-} x_k = \frac{\det(T_1 + y, T_2 + y, \dots, T_{m_-} + y)}{\det T}.$$

First note that

$$(5.63) \quad \det T = \det(1 - \Delta^{-1}A\Delta) = \det(I - A) = \det(I - A^{(n)}).$$

Second, a direct calculation shows that

$$(T_1 + y, T_2 + y, \dots, T_{m_-} + y) = I - A', \quad A'_{jk} = \sum_{\ell=1}^{m_+} \frac{a'_j b'_\ell}{(q_j - p_\ell)(p_\ell - q_k)},$$

where

$$a'_j = a_j q_j^{-1}, \quad b'_\ell = b_\ell p_\ell.$$

Using the definitions (5.43, 5.42) of  $a_j, b_j$ , we note that

$$a'_j = a_j^{(n-1)} + O(n^{-1-\rho}), \quad b'_j = b_j^{(n-1)} + O(n^{-1-\rho}),$$

and therefore  $A' = A^{(n-1)} + O(n^{-1-\rho})$ . Thus we can rewrite (5.62) as

$$(5.64) \quad \widehat{R}_{22}(0) = \Phi_{22}(0) = 1 - \sum_{j=1}^{m_-} \frac{h_j}{q_j} = \frac{\det(I - A^{(n-1)})}{\det(I - A^{(n)})} \left[ 1 + O\left(\frac{1}{n^{1+\rho}}\right) \right].$$

Note that

$$\det(I - A^{(n)}) = 1 + O(n^{-2\rho}).$$

Recalling now (5.51) and the representation of the Toeplitz determinant  $D_n(f)$  as a product of  $\chi_k^{-2}$ , we can, for some sufficiently large  $n_0 > 0$ , using (5.64), write

$$(5.65) \quad D_n(f) = D_{n_0}(f) \prod_{k=n_0+1}^n \chi_{k-1}^{-2} =$$

$$D_{n_0}(f) \prod_{k=n_0+1}^n \left[ 1 + \frac{1}{k} \sum_{j=0}^m (\alpha_j^2 - \beta_j^2) + O\left(\frac{1}{k^{1+\rho}}\right) \right] \frac{\det(I - A^{(k)})}{\det(I - A^{(k-1)})} \left[ 1 + O\left(\frac{1}{k^{1+\rho}}\right) \right] =$$

$$C(n_0, \alpha_0, \dots, \alpha_m, \beta_0, \dots, \beta_m) n^{\sum_{j=0}^m (\alpha_j^2 - \beta_j^2)} \left[ 1 + O\left(\frac{1}{n^\rho}\right) \right],$$

where all the error terms are uniform for  $\alpha_j, \beta_j$  in compact sets, and  $C$  is a constant depending analytically on  $\alpha_j, \beta_j$ , and  $n_0$  only. Recall that in the derivation of this expression we assumed that  $|||\beta||| < 1$  (and as usual  $\Re \alpha_j > -1/2$  for all  $j$ ). Under this condition  $\rho > 0$  (see (5.37)), and the error term tends to zero as  $n \rightarrow \infty$ . In the previous section, we obtained  $C$  explicitly for  $\beta_j$  satisfying  $|||\beta||| < 1/2$ . Obviously, if all  $\alpha_j, \beta_j$  belong to fixed compact sets and the value of  $n_0$  is fixed, the constant  $C$  in (5.65) is bounded above in absolute value by a constant independent of any  $\alpha_j, \beta_j$ . By Vitali's theorem this fact implies that  $C$  can be analytically continued in  $\beta_j$  to the full domain  $|||\beta||| < 1$  off its values on the domain  $|||\beta||| < 1/2$ . Thus  $C$  is given by the same expression also for  $|||\beta||| < 1$ , and this concludes the proof of Theorem 1.1 for  $|||\beta||| < 1$ ,  $V(z) \equiv 0$ , with the error term  $o(1) = O(n^{|||\beta|||-1})$  in (1.10).

**5.3. Adding special analytic  $V(z)$ .** In this section and in the next one, we will add the multiplicative factor  $e^{V(z)}$ , where  $V$  is analytic in a neighborhood of the unit circle  $C$ , to a symbol with pure Fisher-Hartwig singularities and obtain the asymptotics of the corresponding determinant. Consider the deformation of the symbol  $f(z, t)$  given by (3.21) for  $t \in [0, 1]$ . The analysis is based on integration of the differential identity (3.26) over  $t$ . In the present section we assume that  $V$  is such that

$$(5.66) \quad 1 - t + te^{V(z)} \neq 0, \quad t \in [0, 1], \quad z \in C,$$

and  $1 - t + te^{V(z)}$  has no winding around  $C$  for all  $t \in [0, 1]$ . Then the Riemann-Hilbert problem (for the polynomials orthogonal) with  $f(z, t)$  has the same singularities as the problem with  $f(z)$  and is solved in the same way. In the following section we remove the condition (5.66).

Let us rewrite the identity (3.26) in terms of the function  $S(z) \equiv S(z, t)$  which is the solution of the Riemann-Hilbert problem posed on the deformed contour depicted in Figure 1. From the transformation  $Y \rightarrow S$  defined in (4.1), (4.2) we obtain

$$(5.67) \quad Y_{11} = z^n f^{-1} S_{12,+} + S_{11,+}, \quad Y_{21} = z^n f^{-1} S_{22,+} + S_{21,+}, \quad z \in C \equiv \Sigma'.$$

Substituting these expressions into (3.26) and taking into account that  $\det S = 1$  we arrive at the formula:

$$(5.68) \quad \begin{aligned} \frac{\partial}{\partial t} \ln D_n(f(z, t)) &= n \int_C f^{-1} \dot{f} \frac{dz}{2\pi i z} + \int_C [-f^{-2} f' + 2f^{-1} (S'_{22,+} S_{11,+} - S'_{12,+} S_{21,+})] \dot{f} \frac{dz}{2\pi i} \\ &\quad + \int_C [z^{-n} (S'_{21,+} S_{11,+} - S'_{11,+} S_{21,+}) + z^n (S'_{22,+} S_{12,+} - S'_{12,+} S_{22,+}) f^{-2}] \dot{f} \frac{dz}{2\pi i}, \end{aligned}$$

where we introduced the notation  $\dot{f} \equiv \partial f / \partial t$  and  $f' \equiv \partial f / \partial z$ . Using the jump relation (4.4) satisfied by the function  $S(z, t)$  across the unit circle, we can rewrite equation (5.68) in a more symmetric way,

$$(5.69) \quad \begin{aligned} \frac{\partial}{\partial t} \ln D_n(f(z, t)) &= n \int_C f^{-1} \dot{f} \frac{dz}{2\pi i z} + X(t), \\ X(t) &= \int_C [S'_{22,+} S_{11,+} - S'_{12,+} S_{21,+} + S'_{11,-} S_{22,-} - S'_{21,-} S_{12,-}] f^{-1} \dot{f} \frac{dz}{2\pi i} \\ &\quad + \int_C [z^{-n} (S'_{22,-} S_{12,-} - S'_{12,-} S_{22,-}) + z^n (S'_{22,+} S_{12,+} - S'_{12,+} S_{22,+})] f^{-2} \dot{f} \frac{dz}{2\pi i}. \end{aligned}$$

Analytically continuing the boundary values of  $S(z, t)$  from the “+” side of  $C$  to the “−” side of  $\Sigma''$ , and from the “−” side of  $C$  to the “+” side of  $\Sigma$ , we can write the term  $X(t)$  in (5.69) in the form

$$(5.70) \quad X(t) = \int_{\Sigma''} (J_- + z^n I_-) f^{-1} \dot{f} \frac{dz}{2\pi i} + \int_{\Sigma} (-J_+ + z^{-n} I_+) f^{-1} \dot{f} \frac{dz}{2\pi i},$$

where

$$(5.71) \quad I = (S'_{22}S_{12} - S'_{12}S_{22})f^{-1}, \quad J = S'_{22}S_{11} - S'_{12}S_{21}.$$

Denote by  $\Sigma_\varepsilon$  (resp.,  $\Sigma''_\varepsilon$ ) the part of  $\Sigma$  (resp.,  $\Sigma''$ ) which lies inside  $\cup_{j=0}^m U_{z_j}$ . Consider first

$$(5.72) \quad \int_{\Sigma''_\varepsilon} (J_- + z^n I_-) f^{-1} \dot{f} \frac{dz}{2\pi i}.$$

Using (4.3) on  $\Sigma''$  and (5.71), we easily obtain that  $J_- = J_+ - z^n I_+$  and  $I_- = I_+$ , and therefore,

$$(5.73) \quad \int_{\Sigma''_\varepsilon} (J_- + z^n I_-) f^{-1} \dot{f} \frac{dz}{2\pi i} = \int_{\Sigma''_\varepsilon} J_+ f^{-1} \dot{f} \frac{dz}{2\pi i} = \int_{C''_\varepsilon} J f^{-1} \dot{f} \frac{dz}{2\pi i},$$

where  $C''_\varepsilon$  is the part of  $\cup_{j=0}^m \partial U_{z_j}$  lying inside the unit circle from the intersection of  $\partial U_{z_j}$  with the incoming  $\Sigma''$  to the intersection with the outgoing  $\Sigma''$  for each  $j$ . Note that  $J(z, t)$  has no nonintegrable singularity at  $z_j$ . Indeed, for  $z$  inside the smaller sector formed by  $\Sigma''$  at  $z_j$ , we can write

$$(5.74) \quad S(z, t) = Y(z, t) = \widehat{Y}(z, t) \begin{pmatrix} 1 & \kappa(z, t) \\ 0 & 1 \end{pmatrix},$$

where  $\kappa(z, t) = c_1(z, t)(z - z_j)^{2\alpha_j}$ , if  $\alpha_j \neq 0$ , and  $\kappa(z, t) = c_2(z, t) \ln(z - z_j)$ , if  $\alpha_j = 0$ ,  $\beta_j \neq 0$ , for some  $c_k(z, t)$  analytic near  $z_j$ , chosen so that  $\widehat{Y}(z)$  is analytic in a neighborhood of  $z_j$ . Writing  $J$  in terms of the matrix elements of  $\widehat{Y}$ , we see that the contributions of the (singular) derivative  $\kappa'(z)$  cancel, and we obtain

$$(5.75) \quad J = (\widehat{Y}'_{21} \widehat{Y}_{11} - \widehat{Y}_{21} \widehat{Y}'_{11}) \kappa + \widehat{Y}'_{22} \widehat{Y}_{11} - \widehat{Y}'_{12} \widehat{Y}_{21}.$$

We now analyze (5.73) asymptotically. The asymptotic expression for  $S(z, t)$  inside the unit circle and outside  $\cup_{j=0}^m U_{z_j}$  is given by (see (4.26), (4.7))

$$(5.76) \quad S(z, t) = R(z, t) e^{g(z, t)\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where  $g(z)$  is defined by the formula

$$(5.77) \quad \mathcal{D}(z, t) = e^{g(z, t)}.$$

We have

$$J = g' + R'_{11}R_{22} - R_{12}R'_{21} = g' + O_\varepsilon(1/n), \quad n \rightarrow \infty,$$

and we finally obtain

$$(5.78) \quad \int_{\Sigma''_\varepsilon} (J_- + z^n I_-) f^{-1} \dot{f} \frac{dz}{2\pi i} = \int_{C''_\varepsilon} J f^{-1} \dot{f} \frac{dz}{2\pi i} = \int_{C''_\varepsilon} g'(z) f^{-1} \dot{f} \frac{dz}{2\pi i} + O_\varepsilon(1/n), \quad n \rightarrow \infty.$$

Similarly, using the jump condition for  $S$  and then the asymptotics for  $S$  outside the unit circle, we obtain

$$(5.79) \quad \int_{\Sigma_\varepsilon} (-J_+ + z^{-n} I_+) f^{-1} \dot{f} \frac{dz}{2\pi i} = - \int_{C_\varepsilon} J f^{-1} \dot{f} \frac{dz}{2\pi i} = \int_{C_\varepsilon} g'(z) f^{-1} \dot{f} \frac{dz}{2\pi i} + O_\varepsilon(1/n), \quad n \rightarrow \infty,$$

where  $C_\varepsilon$  is the part of  $\cup_{j=0}^m \partial U_{z_j}$  lying outside the unit circle from the intersection of  $\partial U_{z_j}$  with the incoming  $\Sigma$  to the intersection with the outgoing  $\Sigma$  for each  $j$ .

Returning to the integrals (5.70), we now consider the part arising from the integration over  $\Sigma'' \setminus \Sigma''_\varepsilon$  and  $\Sigma \setminus \Sigma_\varepsilon$ . In this part the terms containing  $z^{\pm n} I_\mp$  give a contribution which is exponentially small in  $n$ , while the integration of the terms with  $J$  can be replaced by the integration over  $\Sigma' \setminus \Sigma'_\varepsilon$ ,

where  $\Sigma'_\varepsilon = \Sigma' \cap (\cup_{j=0}^m U_{z_j})$ , and over parts of the boundaries  $\partial U_{z_j}$ . Thus, recalling also (5.78), (5.79), we have

$$(5.80) \quad X(t) = \int_{\Sigma' \setminus \Sigma'_\varepsilon} (J_+ - J_-) f^{-1} \dot{f} \frac{dz}{2\pi i} + \sum_{j=0}^m \left( \int_{\partial U_{z_j}^+} + \int_{\partial U_{z_j}^-} \right) g'(z) f^{-1} \dot{f} \frac{dz}{2\pi i} + O_\varepsilon(1/n), \quad n \rightarrow \infty,$$

where  $\partial U_{z_j}^+$  (resp.,  $\partial U_{z_j}^-$ ) is the part of the boundary of  $U_{z_j}$  inside (resp., outside) the unit circle oriented from the intersection with the incoming  $\Sigma'$  to the intersection with the outgoing  $\Sigma'$ .

Note that using the same considerations as before, we can write in (5.80)  $J_+ - J_- = g'_+ + g'_- + O_\varepsilon(1/n)$ , and therefore

$$(5.81) \quad X(t) = \int_{\Sigma_+} g'_+ f^{-1} \dot{f} \frac{dz}{2\pi i} + \int_{\Sigma_-} g'_- f^{-1} \dot{f} \frac{dz}{2\pi i} + O_\varepsilon(1/n), \quad n \rightarrow \infty,$$

where the closed anticlockwise oriented contours

$$(5.82) \quad \Sigma_+ = (\Sigma' \setminus \Sigma'_\varepsilon) \cup_{j=0}^m \partial U_{z_j}^+, \quad \Sigma_- = (\Sigma' \setminus \Sigma'_\varepsilon) \cup_{j=0}^m \partial U_{z_j}^-.$$

(Note that one can deform  $\Sigma_+$  (resp.,  $\Sigma_-$ ) to a circle around zero of radius  $1 - \varepsilon$  (resp.,  $1 + \varepsilon$ ).)

By (3.21),

$$(5.83) \quad f^{-1} \dot{f} = \frac{-1 + e^{V(z)}}{1 - t + te^{V(z)}} = \frac{\partial}{\partial t} \ln \left( 1 - t + te^{V(z)} \right).$$

Furthermore, writing  $g$  in the form

$$(5.84) \quad g(z, t) = g^{Sz}(z, t) + g^{FH}(z),$$

we have, by (5.77), (4.8), (4.9), (4.10),

$$(5.85) \quad g^{Sz}(z, t) = \int_C \frac{\ln(1 - t + te^{V(s)})}{s - z} \frac{ds}{2\pi i}, \quad g^{FH}(z) = \begin{cases} \sum_{k=1}^m (\alpha_k + \beta_k) \ln \frac{z - z_k}{z_k e^{i\pi}}, & |z| < 1 \\ \sum_{k=1}^m (-\alpha_k + \beta_k) \ln \frac{z - z_k}{z}, & |z| > 1 \end{cases}.$$

Note that the solution of the Riemann-Hilbert problem is uniform in  $t \in [0, 1]$ . From this fact and the explicit formulas (5.83), (5.84, 5.85), we conclude, by an argument similar to the argument following (5.23) that the identity (5.69) holds for all  $t \in [0, 1]$ . Now integrating equation (5.69) from  $t = 0$  to  $t = 1$ , we connect the Toeplitz determinant  $D_n(f(z, 1))$  with the Toeplitz determinant  $D_n(f(z, 0))$  which represents the “pure” Fisher-Hartwig case and whose asymptotics we evaluated in the previous section. First, using (5.83) and changing the order of integration in the first term of (5.69), we obtain

$$(5.86) \quad \int_0^1 dt \int_C f^{-1} \dot{f} \frac{dz}{2\pi i z} = \frac{1}{2\pi} \int_0^{2\pi} V(e^{i\theta}) d\theta = V_0.$$

Furthermore, by (5.81), (5.84, 5.85),

$$(5.87) \quad \int_0^1 dt X(t) = I^{Sz} + I^{FH} + O_\varepsilon(1/n), \quad n \rightarrow \infty,$$

where (cf. [17], Eq (86), (87))

$$(5.88) \quad I^{Sz} = \int_0^1 dt \int_C ((g^{Sz})'_+ + (g^{Sz})'_-) f^{-1} \dot{f} \frac{dz}{2\pi i} = \sum_{k=1}^\infty k V_k V_{-k},$$

and

$$(5.89) \quad I^{FH} = \int_{\Sigma_+} (g^{FH}(z))'_+ V(z) \frac{dz}{2\pi i} + \int_{\Sigma_-} (g^{FH}(z))'_- V(z) \frac{dz}{2\pi i} \\ = \sum_{k=0}^m \left[ (\alpha_k + \beta_k) \int_{\Sigma_+} \frac{V(z)}{z - z_k} \frac{dz}{2\pi i} + (-\alpha_k + \beta_k) \int_{\Sigma_-} \left( \frac{V(z)}{z - z_k} - \frac{V(z)}{z} \right) \frac{dz}{2\pi i} \right].$$

Since

$$(5.90) \quad g^{Sz}(z, 1) = \int_C \frac{V(s)}{s - z} \frac{ds}{2\pi i} = \begin{cases} \ln b_+(z) + V_0, & |z| < 1 \\ -\ln b_-(z), & |z| > 1 \end{cases},$$

we obtain

$$(5.91) \quad \ln b_+(z_k) = \int_{\Sigma_-} \frac{V(z)}{z - z_k} \frac{dz}{2\pi i} - V_0, \quad \ln b_-(z_k) = - \int_{\Sigma_+} \frac{V(z)}{z - z_k} \frac{dz}{2\pi i},$$

which finally gives

$$(5.92) \quad I^{FH} = \sum_{k=0}^m [-(\alpha_k + \beta_k) \ln b_-(z_k) + (-\alpha_k + \beta_k) \ln b_+(z_k)].$$

Collecting (5.86), (5.87), (5.88), and (5.92), we obtain from (5.69)

$$(5.93) \quad \ln D_n(f(z, 1)) - \ln D_n(f(z, 0)) = nV_0 + \sum_{k=1}^{\infty} kV_k V_{-k} \\ + \sum_{k=0}^m [-(\alpha_k + \beta_k) \ln b_-(z_k) + (-\alpha_k + \beta_k) \ln b_+(z_k)] + O_\varepsilon(1/n), \quad n \rightarrow \infty,$$

which, in view of the result of the previous section, concludes the proof of Theorem 1.1 for analytic  $V(z)$  satisfying the condition (5.66).

**5.4. Extension to general analytic  $V(z)$ .** Now let  $V(z)$  be any function analytic in a neighborhood of the unite circle. Since zeros of the expression  $1 - t + te^{V(z)}$ ,  $t \in [0, 1]$ ,  $z \in C$ , can only occur if  $\Im V(z) = \pi(2k + 1)$ ,  $k \in \mathbb{Z}$ , there exists a positive integer  $q$  such that  $\frac{1}{q}V(z)$  satisfies the condition (5.66) of the previous section, i.e.,

$$1 - t + te^{\frac{1}{q}V(z)} \neq 0, \quad \forall t \in [0, 1], \quad z \in C,$$

and this function has no winding around  $C$  for all  $t \in [0, 1]$ . Let

$$(5.94) \quad f_0(z) = f^{FH}(z), \quad f_\ell(z) = e^{\frac{1}{q}V(z)} f_{\ell-1}(z), \quad \ell = 1, \dots, q,$$

where  $f^{FH}(z)$  is the symbol for the “pure” Fisher-Hartwig case. Note that

$$(5.95) \quad f(z) \equiv e^{V(z)} f^{FH}(z) = f_q(z).$$

Consider  $f_\ell(z)$ ,  $\ell = 1, \dots, q$ , and introduce the deformation,

$$(5.96) \quad f_\ell(z, t) = \left(1 - t + te^{\frac{1}{q}V(z)}\right) f_{\ell-1}(z) = \left(1 - t + te^{\frac{1}{q}V(z)}\right) e^{\frac{\ell-1}{q}V(z)} f^{FH}(z).$$

All the considerations of the part of the previous section between equations (5.67) and (5.81) go through with  $f(z, t)$  replaced by  $f_\ell(z, t)$  and we arrive at the formulae:

$$(5.97) \quad \frac{\partial}{\partial t} \ln D_n(f_\ell(z, t)) = n \int_C f_\ell^{-1} \dot{f}_\ell \frac{dz}{2\pi i z} + X(t),$$

$$X(t) = \int_{\Sigma_+} g'_+ f_\ell^{-1} \dot{f}_\ell \frac{dz}{2\pi i} + \int_{\Sigma_-} g'_- f_\ell^{-1} \dot{f}_\ell \frac{dz}{2\pi i} + O_\varepsilon(1/n), \quad n \rightarrow \infty,$$

where, as before, the closed anticlockwise oriented contours

$$(5.98) \quad \Sigma_+ = (\Sigma' \setminus \Sigma'_\varepsilon) \cup_{j=0}^m \partial U_{z_j}^+, \quad \Sigma_- = (\Sigma' \setminus \Sigma'_\varepsilon) \cup_{j=0}^m \partial U_{z_j}^-,$$

and  $g(z) \equiv g_\ell(z)$  now corresponds to  $f_\ell$ .

The first term in (5.97) yields (cf. (5.86))

$$(5.99) \quad \int_0^1 dt \int_C f_\ell^{-1} \dot{f}_\ell \frac{dz}{2\pi i z} = \frac{1}{2\pi q} \int_0^{2\pi} V(e^{i\theta}) d\theta = \frac{1}{q} V_0.$$

In order to evaluate  $X(t)$ , we write  $g$  in the form

$$(5.100) \quad g(z, t) = g^{Sz}(z, t) + \tilde{g}^{Sz}(z) + g^{FH}(z),$$

where  $g^{FH}(z)$  is the same as in (5.85), and

$$(5.101) \quad g^{Sz}(z, t) = \int_C \frac{\ln(1 - t + te^{\frac{1}{q}V(s)})}{s - z} \frac{ds}{2\pi i}, \quad \tilde{g}^{Sz}(z) = \frac{\ell - 1}{q} \int_C \frac{V(s)}{s - z} \frac{ds}{2\pi i}.$$

Then we obtain

$$(5.102) \quad \int_0^1 dt X(t) = I^{Sz} + \tilde{I}^{Sz} + I^{FH} + O_\varepsilon(1/n), \quad n \rightarrow \infty,$$

where, up to the replacement  $V \rightarrow \frac{1}{q}V$ , the integrals  $I^{Sz}$  and  $I^{FH}$  are the respective integrals from the previous section, i.e.,

$$(5.103) \quad I^{Sz} = \frac{1}{q^2} \sum_{k=1}^{\infty} k V_k V_{-k}, \quad I^{FH} = \frac{1}{q} \sum_{k=0}^m [-(\alpha_k + \beta_k) \ln b_-(z_k) + (-\alpha_k + \beta_k) \ln b_+(z_k)].$$

The term  $\tilde{I}^{Sz}$  in (5.102) is given by the equation

$$(5.104) \quad \tilde{I}^{Sz} = \frac{1}{q} \int_C \left( (\tilde{g}^{Sz}(z))'_+ + (\tilde{g}^{Sz}(z))'_- \right) V(z) \frac{dz}{2\pi i}.$$

Note that

$$(\tilde{g}^{Sz}(z))'_+ = \frac{\ell - 1}{q} \sum_{k=1}^{\infty} k z^{k-1} V_k, \quad (\tilde{g}^{Sz}(z))'_- = \frac{\ell - 1}{q} \sum_{k=1}^{\infty} k z^{-k-1} V_{-k}.$$

Therefore, after a simple calculation we obtain

$$(5.105) \quad \tilde{I}^{Sz} = \frac{2\ell - 2}{q^2} \sum_{k=1}^{\infty} k V_k V_{-k}.$$

Integrating (5.97) from  $t = 0$  to  $t = 1$  and taking into account (5.99), (5.102), (5.103), and (5.105), we obtain the following equation for the determinant  $D_n(f_\ell(z))$ :

$$(5.106) \quad \ln D_n(f_\ell(z)) - \ln D_n(f_{\ell-1}(z)) = \frac{1}{q}nV_0 + \frac{2\ell-1}{q^2} \sum_{k=1}^{\infty} kV_kV_{-k} \\ + \frac{1}{q} \sum_{k=0}^m [-(\alpha_k + \beta_k) \ln b_-(z_k) + (-\alpha_k + \beta_k) \ln b_+(z_k)] + O_\varepsilon(1/n), \quad n \rightarrow \infty.$$

This equation holds for any  $\ell = 1, \dots, q$ . Summing up from  $\ell = 1$  to  $\ell = q$  we again arrive at the formula

$$(5.107) \quad \ln D_n(f(z)) - \ln D_n(f^{FH}(z)) = nV_0 + \sum_{k=1}^{\infty} kV_kV_{-k} \\ + \sum_{k=0}^m [-(\alpha_k + \beta_k) \ln b_-(z_k) + (-\alpha_k + \beta_k) \ln b_+(z_k)] + O_\varepsilon(1/n), \quad n \rightarrow \infty,$$

which concludes the proof of Theorem 1.1 in the case of  $V(z)$  analytic in a neighborhood of the unit circle.

**5.5. Extension to smooth  $V(z)$ .** If  $V(z)$  is just sufficiently smooth, in particular  $C^\infty$ , on the unit circle  $C$  so that (1.11) holds for  $s$  from zero up to and including some  $s \geq 0$ , we can approximate  $V(z)$  by trigonometric polynomials  $V^{(n)}(z) = \sum_{k=-p(n)}^{p(n)} V_k z^k$ ,  $z \in C$ . First, consider the case when  $|||\beta||| = \max_{j,k} |\Re \beta_j - \Re \beta_k| = 2 \max_j |\Re \beta_j - \omega| < 1$ , where  $\omega$  is defined by (4.31). (The indices  $j, k = 0$  are omitted if  $\alpha_0 = \beta_0 = 0$ .) We set

$$(5.108) \quad p = [n^{1-\nu}], \quad \nu = |||\beta||| + \varepsilon_1,$$

where  $\varepsilon_1 > 0$  is chosen sufficiently small so that  $\nu < 1$  (square brackets denote the integer part).

First, we need to extend the RH analysis of the previous sections to symbols which depend on  $n$ , namely to the case when  $V$  in  $f$  is replaced by  $V^{(n)}$ . (We will denote such  $f$  by  $f(z, V^{(n)})$ , and the original  $f$  by  $f(z, V)$ .) We need to have a suitable estimate for the behavior of the error term in the asymptotics with  $n$ . For a fixed  $f$ , our analysis depended, in particular, on the fact that  $f(z)^{-1}z^{-n}$  is of order  $e^{-\varepsilon'n}$ ,  $\varepsilon' > 0$ , for  $z \in \Sigma^{\text{out}}$  (see Section 4.1), and similarly,  $f(z)^{-1}z^n = O(e^{-\varepsilon'n})$  for  $z \in \Sigma''^{\text{out}}$ . Here the contours  $\Sigma^{\text{out}}$ ,  $\Sigma''^{\text{out}}$  are outside a *fixed* neighborhood of the unit circle (outside and inside  $C$ , respectively). If  $V$  is replaced by  $V^{(n)}$ , let us define the curve  $\Sigma$  outside  $\cup_{j=0}^m U_{z_j}$  by

$$(5.109) \quad z = \left(1 + \gamma \frac{\ln p}{p}\right) e^{i\theta}, \quad \gamma > 0,$$

and  $\Sigma''$  outside  $\cup_{j=0}^m U_{z_j}$  by

$$(5.110) \quad z = \left(1 - \gamma \frac{\ln p}{p}\right) e^{i\theta}.$$

Inside all the sets  $U_{z_j}$ , the curves still go to  $z_j$  as discussed in Section 4. Let the radius of all  $U_{z_j}$  be  $2\gamma \ln p/p$ . We now fix the value of  $\gamma$  as follows. Using the condition (1.11) we can write (here

and below  $c$  stands for various positive constants independent of  $n$ )

(5.111)

$$\begin{aligned} |V^{(n)}(z)| - |V_0| &\leq \sum_{k=-p, k \neq 0}^p |k^s V_k| \frac{|z|^k}{|k|^s} < c \left( \sum_{k=-p, k \neq 0}^p |k^s V_k|^2 \right)^{1/2} \left( \sum_{k=1}^p \frac{(1 \pm 3\gamma \ln p/p)^{\pm 2k}}{k^{2s}} \right)^{1/2} \\ &< c \left( \sum_{k=1}^p \frac{(1 \pm 3\gamma \ln k/k)^{\pm 2k}}{k^{2s}} \right)^{1/2} < c \left( \sum_{k=1}^p \frac{1}{k^{2(s-3\gamma)}} \left[ 1 + O\left(\frac{\ln^2 k}{k}\right) \right] \right)^{1/2}, \end{aligned}$$

where  $z \in \Sigma^{\text{out}}$ ,  $z \in \partial U_{z_j} \cap \{|z| > 1\}$  (with “+” sign in “ $\pm$ ”), and  $z \in \Sigma''^{\text{out}}$ ,  $z \in \partial U_{z_j} \cap \{|z| < 1\}$  (with “−” sign). We now set

$$(5.112) \quad 3\gamma = s - (1 + \varepsilon_2)/2, \quad \varepsilon_2 > 0,$$

and then

$$(5.113) \quad |V^{(n)}(z)| < c, \quad |b_+(z, V^{(n)})| < c, \quad |b_-(z, V^{(n)})| < c, \quad \text{for all } n$$

uniformly on  $\Sigma^{\text{out}}$ ,  $\Sigma''^{\text{out}}$ ,  $\partial U_{z_j}$ ’s, and in fact in the whole annulus  $1 - 3\gamma \frac{\ln p}{p} < |z| < 1 + 3\gamma \frac{\ln p}{p}$ .

It is easy to adapt the considerations of the previous sections to the present case, and we again obtain the expansion (4.19) for the jump matrix of  $R$  on  $\partial U_{z_j}$ . Note that now  $|\zeta(z)| = O(n^\nu \ln n)$  and  $|z - z_j| = \ln n / n^{1-\nu}$  as  $n \rightarrow \infty$  for  $z \in \partial U_{z_j}$ , and therefore using (4.19), (4.13), (3.13), (4.9) and the definition of  $\nu$  in (5.108), we obtain, in particular,

$$(5.114) \quad n^{\omega\sigma_3} \Delta_1(z) n^{-\omega\sigma_3} = O\left(\frac{1}{n^{\varepsilon_1} \ln n}\right), \quad z \in \cup_{j=0}^m \partial U_{z_j}.$$

Furthermore, as follows from (5.109), (5.110), (5.113), and (4.27,4.28), the jump matrix on  $\Sigma^{\text{out}}$  and  $\Sigma''^{\text{out}}$  is now the identity plus a function uniformly bounded in absolute value by

$$(5.115) \quad c \left( \frac{n^{1-\nu}}{\ln n} \right)^{2 \max_j |\Re \beta_j|} \left( 1 \pm \gamma(1-\nu) \frac{\ln n}{n^{1-\nu}} \right)^{\mp n} < c \exp \left\{ -\frac{\gamma}{2} (1-\nu) n^\nu \ln n \right\} n^{2(1-\nu) \max_j |\Re \beta_j|},$$

where the upper sign corresponds to  $\Sigma^{\text{out}}$ , and the lower to  $\Sigma''^{\text{out}}$ .

The RH problem for  $R(z)$  (see Section 4.1) is therefore solvable, and we obtain  $R(z)$  as a series where the first term  $R_1$  is the same as before, and for the error term the same estimate holds for  $z$  outside a fixed neighborhood of the unit circle, e.g., for  $z$  large.

This implies that Theorem 1.1 holds for  $f(z, V^{(n)})$ . Note that it also holds for  $|f(z, V^{(n)})|$ .

We will now show that replacing  $V^{(n)}$  with  $V$  in the symbol of the determinant  $D_n(f(z, V^{(n)}))$  results, under a condition on  $s$ , in a small error only, so that Theorem 1.1 holds for  $D_n(f(z, V))$  as well.

Using the Heine representation (2.10) for a Toeplitz determinant with (any) symbol  $f(z)$ , the straightforward estimate

$$(5.116) \quad b_\pm(z, V^{(n)}) = b_\pm(z, V) \left[ 1 + O\left(\frac{1}{n^{(1-\nu)s}}\right) \right], \quad \text{uniformly for } |z| = 1,$$



which follows from (1.11), and Theorem 1.1 for  $D_n(|f(z, V^{(n)})|)$  and  $D_n(f(z, V^{(n)}))$ , we have if  $s(1 - \nu) > 1$ ,

$$\begin{aligned}
 (5.117) \quad & \left| D_n(f(z, V)) - D_n(f(z, V^{(n)})) \right| < \\
 & \frac{1}{(2\pi)^n n!} \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{1 \leq j < k \leq n} |e^{i\phi_j} - e^{i\phi_k}|^2 \prod_{j=0}^n |f(e^{i\phi_j}, V^{(n)})| d\phi_j \times \left( \left| 1 + c/n^{(1-\nu)s} \right|^n - 1 \right) \\
 & < c e^{\Re V_0 n \sum_{j=0}^m ((\Re \alpha_j)^2 + (\Im \beta_j)^2)} (e^{c/n^{(1-\nu)s-1}} - 1) \\
 & < c \left| e^{V_0 n \sum_{j=0}^m (\alpha_j^2 - \beta_j^2)} \right| n^{\sum_{j=0}^m ((\Im \alpha_j)^2 + (\Re \beta_j)^2)} \frac{1}{n^{(1-\nu)s-1}} \\
 & < c \left| D_n(f(z, V^{(n)})) \right| n^{-((1-\nu)s-1 - \sum_{j=0}^m ((\Im \alpha_j)^2 + (\Re \beta_j)^2))}.
 \end{aligned}$$

Therefore,

$$(5.118) \quad D_n(f(z, V)) = D_n(f(z, V^{(n)})) \left( 1 + \frac{D_n(f(z, V)) - D_n(f(z, V^{(n)}))}{D_n(f(z, V^{(n)}))} \right) = D_n(f(z, V^{(n)}))(1 + o(1)),$$

if

$$(5.119) \quad s > \frac{1 + \sum_{j=0}^m ((\Im \alpha_j)^2 + (\Re \beta_j)^2)}{1 - \nu}.$$

Note that this condition is consistent with (5.112) and the requirement that  $\gamma > 0$ . Using the expression for  $\nu$  in (5.108) and noting that  $\varepsilon_1$  can be arbitrary close to zero, we replace (5.119) with (1.12). Under the condition (1.12) we then obtain the statement of the theorem for  $D_n(f(z, V))$ .

## 6. APPENDIX. THE TOEPLITZ DETERMINANT $D_n$ AS A TAU-FUNCTION

In this section we construct a Fuchsian system of ODE's corresponding to the Riemann-Hilbert problem of Section 2 for  $V \equiv 0$ . We show that the differential identities (3.20) for the Toeplitz determinant can be viewed as monodromy deformations of the tau-function associated with this Fuchsian system.

Assume the pure Fisher-Hartwig case,  $V(z) \equiv 0$ . Set

$$(6.1) \quad \Phi(z) = \Lambda Y^{(n)}(z) \Lambda^{-1} \prod_{k=0}^m (z - z_k)^{\alpha_k \sigma_3} z^{\lambda \sigma_3},$$

where

$$\Lambda = \prod_{k=0}^m z_k^{\frac{\beta_k + \alpha_k}{2} \sigma_3}, \quad \lambda = \sum_{k=0}^m \frac{\beta_k - \alpha_k}{2} - \frac{n}{2},$$

and the branches of all multi-valued functions are chosen as in Section 3. In terms of the function  $\Phi(z)$ , the Riemann-Hilbert problem (2.6)- (2.7) reads as follows:

- (a)  $\Phi(z)$  is analytic for  $z \in \mathbb{C} \setminus (C \cup [0, 1] \cup \{\cup_{j=0}^m \Gamma_j\})$ , where  $\Gamma_j$  is the ray  $\theta = \theta_j$  from  $z_j$  to infinity. The unit circle  $C$  is oriented as before, counterclockwise, the segment  $[0, 1]$  is oriented from 0 to 1, and the rays  $\Gamma_j$  are oriented towards infinity.

(b) The boundary values of  $\Phi(z)$  are related by the jump conditions,

$$(6.2) \quad \Phi_+(z) = \Phi_-(z) \begin{cases} \begin{pmatrix} 1 & s_j \\ 0 & 1 \end{pmatrix}, & z \in C, \quad \theta_j < \arg z < \theta_{j+1}, \quad 0 \leq j \leq m, \quad \theta_{m+1} = 2\pi \\ e^{-2\pi i \alpha_j \sigma_3}, & z \in \Gamma_j, \quad 1 \leq j \leq m \\ e^{-2\pi i (\alpha_0 + \lambda) \sigma_3}, & z \in \Gamma_0 \\ e^{-2\pi i \lambda \sigma_3}, & z \in [0, 1] \end{cases}$$

where

$$(6.3) \quad s_j = \exp \left\{ -i\pi \sum_{k=0}^j \beta_k + i\pi \sum_{k=j+1}^m \beta_k - i\pi \sum_{k=0}^j \alpha_k - 3i\pi \sum_{k=j+1}^m \alpha_k \right\}.$$

(for  $j = m$ , the second and the fourth sums are absent)

(c)  $\Phi(z)$  has the following asymptotic behavior at infinity:

$$(6.4) \quad \Phi(z) = \left( I + O\left(\frac{1}{z}\right) \right) z^{\left(\sum_{k=0}^m \frac{\beta_k + \alpha_k}{2} + \frac{n}{2}\right) \sigma_3} e^{2\pi i \sum_{k=j+1}^m \alpha_k \sigma_3},$$

as  $z \rightarrow \infty$  and  $\theta_j < \arg z < \theta_{j+1}$ ,  $0 \leq j \leq m$ ,  $\theta_{m+1} = 2\pi$  (for  $j = m$  the last factor is omitted).

(d) In the neighborhoods  $U_{z_j}$  of the points  $z_j$ ,  $j = 0, 1, \dots, m$ , the function  $\Phi(z)$  admits the following representations, which constitute a refinement of the estimates (2.8) and (2.9).

- If  $\alpha_j \neq 0$ , then

$$(6.5) \quad \Phi(z) = \tilde{\Phi}_j(z) (z - z_j)^{\alpha_j \sigma_3} C_j,$$

where  $\tilde{\Phi}_j(z)$  is holomorphic at  $z = z_j$  (it is essentially the function  $\tilde{Y}(z)$  from Section 3) and the matrix  $C_j$  is given by the formula,

$$(6.6) \quad C_j = \begin{pmatrix} 1 & c_j \\ 0 & 1 \end{pmatrix},$$

with

$$(6.7) \quad c_j = s_j \begin{cases} \frac{1 - e^{2\pi i (\beta_j + \alpha_j)}}{1 - e^{4\pi i \alpha_j}}, & z \in U_{z_j}, \quad |z| < 1 \\ \frac{1 - e^{2\pi i (\beta_j - \alpha_j)}}{1 - e^{4\pi i \alpha_j}}, & z \in U_{z_j}, \quad |z| > 1, \quad \arg z < \theta_j \\ e^{4\pi i \alpha_j} \frac{1 - e^{2\pi i (\beta_j - \alpha_j)}}{1 - e^{4\pi i \alpha_j}}, & z \in U_{z_j}, \quad |z| > 1, \quad \arg z > \theta_j \end{cases}$$

in the case  $j \neq 0$ , and

$$(6.8) \quad C_0 = \begin{pmatrix} 1 & c_0 \\ 0 & 1 \end{pmatrix} \times \begin{cases} I, & \Im z > 0 \\ e^{2\pi i \kappa \sigma_3}, & \Im z < 0 \end{cases}$$

with

$$(6.9) \quad c_0 = s_0 \begin{cases} \frac{1-e^{2\pi i(\beta_0+\alpha_0)}}{1-e^{4\pi i\alpha_0}}, & z \in U_{z_0}, \quad |z| < 1 \\ \frac{1-e^{2\pi i(\beta_0-\alpha_0)}}{1-e^{4\pi i\alpha_0}}, & z \in U_{z_0}, \quad |z| > 1, \quad \Im z < 0 \\ e^{4\pi i\alpha_0} \frac{1-e^{2\pi i(\beta_0-\alpha_0)}}{1-e^{4\pi i\alpha_0}}, & z \in U_{z_0}, \quad |z| > 1, \quad \Im z > 0 \end{cases}$$

in the case  $j = 0$ .

- If  $\alpha_j = 0$  and  $\beta_j \neq 0$ , then

$$(6.10) \quad \Phi(z) = \tilde{\Phi}_j(z) \begin{pmatrix} 1 & \frac{d_j}{2\pi i} \ln(z - z_j) \\ 0 & 1 \end{pmatrix} C_j, \quad d_j = -s_j(1 - e^{2\pi i\beta_j}),$$

where  $\tilde{\Phi}_j(z)$  is again holomorphic at  $z = z_j$  and the matrix  $C_j$  this time is given by the formula

$$(6.11) \quad C_j = \begin{pmatrix} 1 & c_j \\ 0 & 1 \end{pmatrix},$$

with

$$(6.12) \quad c_j = \begin{cases} s_{j-1}, & z \in U_{z_j}, \quad |z| < 1 \\ 0 & z \in U_{z_j}, \quad |z| > 1, \quad \arg z < \theta_j \\ d_j, & z \in U_{z_j}, \quad |z| > 1, \quad \arg z > \theta_j \end{cases}$$

in the case  $j \neq 0$ , and

$$(6.13) \quad C_0 = \begin{pmatrix} 1 & c_0 \\ 0 & 1 \end{pmatrix} \times \begin{cases} I, & \Im z > 0 \\ e^{2\pi i\lambda\sigma_3}, & \Im z < 0 \end{cases}$$

with

$$(6.14) \quad c_0 = \begin{cases} s_m e^{4\pi i\lambda}, & z \in U_{z_0}, \quad |z| < 1 \\ 0 & z \in U_{z_0}, \quad |z| > 1, \quad \Im z < 0, \\ d_0, & z \in U_{z_0}, \quad |z| > 1, \quad \Im z > 0 \end{cases}$$

in the case  $j = 0$ .

- (e) In a small neighborhood of  $z = 0$ , the function  $\Phi(z)$  admits a similar representation:

$$(6.15) \quad \Phi(z) = \tilde{\Phi}^{(0)}(z) z^{\lambda\sigma_3},$$

where  $\tilde{\Phi}^{(0)}(z)$  is holomorphic at  $z = 0$ .

We also note that all the matrices above have determinants equal to 1.

A key feature of the  $\Phi$ -RH problem is that all its jump matrices and the connection matrices  $C_j$  are piecewise constant in  $z$ . By standard arguments (see, e.g., [24], [23]), based on the Liouville theorem, this fact implies that the function  $\Phi(z)$  satisfies a linear matrix ODE of Fuchsian type:

$$(6.16) \quad \frac{d\Phi(z)}{dz} = A(z)\Phi(z), \quad A(z) = \sum_{k=0}^m \frac{A_k}{z - z_k} + \frac{B}{z}$$

with

$$(6.17) \quad B = \lambda \tilde{\Phi}^{(0)}(0) \sigma_3 \left( \tilde{\Phi}^{(0)}(0) \right)^{-1}$$

and

$$(6.18) \quad A_j = \begin{cases} \alpha_j \tilde{\Phi}_j(z_j) \sigma_3 \tilde{\Phi}_j^{-1}(z_j), & \text{if } \alpha_j \neq 0 \\ \frac{d_j}{2\pi i} \tilde{\Phi}_j(z_j) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tilde{\Phi}_j^{-1}(z_j), & \text{if } \alpha_j = 0, \quad \beta_j \neq 0. \end{cases}$$

Moreover, since the jump matrices and the connection matrices  $C_j$  are all constant with respect to  $z_j$ , the function  $\Phi(z)$  satisfies, in addition to (6.16), the equations

$$(6.19) \quad \frac{\partial \Phi(z)}{\partial z_j} = -\frac{A_j}{z - z_j} \Phi(z), \quad j = 1, \dots, m.$$

The compatibility condition of (6.16) and (6.19) yields the following nonlinear systems of ODEs on the matrix coefficients  $B$  and  $A_j$ :

$$(6.20) \quad \frac{\partial B}{\partial z_j} = \frac{[A_j, B]}{z_j}, \quad \frac{\partial A_k}{\partial z_j} = \frac{[A_j, A_k]}{z_j - z_k}, \quad k \neq j$$

$$(6.21) \quad \frac{\partial A_j}{\partial z_j} = -\sum_{\substack{k=0 \\ k \neq j}}^m \frac{[A_j, A_k]}{z_j - z_k} + \frac{[B, A_j]}{z_j}.$$

In the context of the Fuchsian system (6.16), the jump matrices and the connection matrices  $C_j$  of the  $\Phi$ -Riemann-Hilbert problem form the monodromy data of the system. The fact that these data do not depend on the parameters  $z_j$  means that the functions  $B \equiv B(z_1, \dots, z_m)$  and  $A_j \equiv A_j(z_1, \dots, z_m)$ ,  $j = 0, \dots, m$  describe *isomonodromy deformations* of the system (6.16). Equations (6.20)–(6.21) are the classical Schlesinger equations.

An important role in the modern theory of isomonodromy deformations is played by the notion of a  $\tau$ -function which was introduced by M. Jimbo, T. Miwa and K. Ueno in [24]. In the Fuchsian case, the  $\tau$ -function is defined as follows. Let

$$(6.22) \quad \tilde{\omega} = \sum_{k=1}^m \text{Res}_{z=z_k} \text{trace } A(z) \frac{\partial \tilde{\Phi}_k(z)}{\partial z} \tilde{\Phi}_k^{-1}(z) dz_k = \sum_{k=1}^m \text{trace } A_k \frac{\partial \tilde{\Phi}_k(z_k)}{\partial z} \tilde{\Phi}_k^{-1}(z_k) dz_k.$$

As shown in [24], the differential form  $\tilde{\omega} \equiv \tilde{\omega}(A_0, \dots, A_m, B; z_1, \dots, z_m)$  is closed on the solutions of the Schlesinger system (6.20)–(6.21). The  $\tau$ -function is then defined as the exponential of the antiderivative of  $\tilde{\omega}$ , i.e.,

$$(6.23) \quad \frac{\partial \ln \tau}{\partial z_k} = \text{trace } A_k \frac{\partial \tilde{\Phi}_k(z_k)}{\partial z} \tilde{\Phi}_k^{-1}(z_k).$$

It was already observed (see, e.g., [23] and [6]) that the  $\tau$ -functions evaluated on solutions of the Schlesinger equations generated by the Riemann-Hilbert problems associated with Toeplitz and Hankel determinants coincide, up to trivial factors, with the determinants themselves. In particular, for the Toeplitz determinant  $D_n$  with the pure Fisher-Hartwig symbol,  $V(z) \equiv 0$ , one can follow the calculations of [23] and obtain that

$$(6.24) \quad \frac{\partial \ln D_n}{\partial z_k} = \text{trace } A_k \frac{\partial \tilde{\Phi}_k(z_k)}{\partial z} \tilde{\Phi}_k^{-1}(z_k) + 2 \sum_{\substack{j=0 \\ j \neq k}}^m \frac{\alpha_k \alpha_j}{z_j - z_k} - 2\lambda \frac{\alpha_k}{z_k}.$$

Therefore, in the case of the Riemann-Hilbert problem (6.2)–(6.15), the relation between the associated Toeplitz determinant and the  $\tau$ -function is given by

$$(6.25) \quad D_n(f(z)) = D_n(z_1, \dots, z_m | \alpha, \beta) = \tau(z_1, \dots, z_m | \alpha, \beta) \prod_{j < k} (z_k - z_j)^{-2\alpha_j \alpha_k} \prod_{j=0}^m z_j^{-2\alpha_j \lambda}.$$

A direct substitution of formulae (6.5), (6.10), and (6.15) into equations (6.16) and (6.19) yields

$$(6.26) \quad \text{trace } A_k \frac{\partial \tilde{\Phi}_k(z_k)}{\partial z} \tilde{\Phi}_k^{-1}(z_k) = \sum_{\substack{j=0 \\ j \neq k}}^m \frac{\text{trace } A_j A_k}{z_k - z_j} + \frac{1}{z_k} \text{trace } B A_k \\ = \sum_{j=0}^m \text{trace } A_j \frac{\partial \tilde{\Phi}_j(z_j)}{\partial z_k} \tilde{\Phi}_j^{-1}(z_j) + \text{trace } B \frac{\partial \tilde{\Phi}^{(0)}(0)}{\partial z_k} \left( \tilde{\Phi}^{(0)}(0) \right)^{-1}.$$

Hence equation (6.24) can be written as follows:

**Lemma 6.1.** *Let the Riemann-Hilbert problem for  $\Phi$  be solvable. For any  $k = 0, 1, \dots, m$ ,*

$$(6.27) \quad \frac{\partial \ln D_n}{\partial z_k} = \sum_{j=0}^m \text{trace } A_j \frac{\partial \tilde{\Phi}_j(z_j)}{\partial z_k} \tilde{\Phi}_j^{-1}(z_j) + \text{trace } B \frac{\partial \tilde{\Phi}^{(0)}(0)}{\partial z_k} \left( \tilde{\Phi}^{(0)}(0) \right)^{-1} + 2 \sum_{\substack{j=0 \\ j \neq k}}^m \frac{\alpha_k \alpha_j}{z_j - z_k} - 2\lambda \frac{\alpha_k}{z_k}.$$

What is of interest here is that the differential identities (3.20), which play a very important role in the main text, can be written in a matrix form closely related to (6.27). For simplicity, we present only the case  $\alpha_j \neq 0$ . We have

**Lemma 6.2.** *Let the Riemann-Hilbert problem for  $\Phi$  be solvable. Let  $\alpha_j \neq 0$ ,  $j = 0, \dots, m$ . Then for any  $k = 0, 1, \dots, m$ ,*

$$(6.28) \quad \frac{\partial \ln D_n}{\partial \alpha_k} = \sum_{j=0}^m \text{trace } A_j \frac{\partial \tilde{\Phi}_j(z_j)}{\partial \alpha_k} \tilde{\Phi}_j^{-1}(z_j) + \text{trace } B \frac{\partial \tilde{\Phi}^{(0)}(0)}{\partial \alpha_k} \left( \tilde{\Phi}^{(0)}(0) \right)^{-1} \\ - 2 \sum_{\substack{j=0 \\ j \neq k}}^m \alpha_j \ln(z_j - z_k) - 2\lambda \ln(-z_k) + \sum_{j=0}^m \alpha_j \ln z_j - n \ln z_k,$$

$$(6.29) \quad \frac{\partial \ln D_n}{\partial \beta_k} = \sum_{j=0}^m \text{trace } A_j \frac{\partial \tilde{\Phi}_j(z_j)}{\partial \beta_k} \tilde{\Phi}_j^{-1}(z_j) + \text{trace } B \frac{\partial \tilde{\Phi}^{(0)}(0)}{\partial \beta_k} \left( \tilde{\Phi}^{(0)}(0) \right)^{-1} - \sum_{j=0}^m \alpha_j \ln z_j - n \ln z_k.$$

*Remark 6.1.* The significance of these equations is that they complement the isomonodromy deformation formula (6.27) by formulae which describe the *monodromy* deformations of the  $\tau$ -function (represented by the Toeplitz determinants  $D_n$ ). (Equations (6.28) and (6.29) should be compared with the general constructions of the recent paper [7].)

*Proof.* Although straightforward, it is rather tedious to derive (6.28) and (6.29) from the differential identities (3.20). There is, however, an alternative way to obtain (6.28) and (6.29) based on the direct analysis of the Riemann-Hilbert problem (2.6)–(2.7). First, we observe that the basic deformation formula for the Toeplitz determinant  $D_n$ , i.e. equations (3.4), (3.5) can be written, using Eq. (2.4) in [15], in the form

$$(6.30) \quad \frac{\partial}{\partial \gamma} \ln D_n = \frac{1}{2\pi i} \int_C z^{-n} \left( Y_{11}(z) \frac{dY_{21}(z)}{dz} - \frac{dY_{11}(z)}{dz} Y_{21}(z) \right) \frac{\partial f(z)}{\partial \gamma} dz.$$

Here, as in Section 3,  $\gamma$  is either  $\alpha_k$  of  $\beta_k$ . Second, by  $\gamma$ -differentiating the Riemann-Hilbert problem (2.6)–(2.7) we easily obtain the following representation for the logarithmic derivative  $\frac{\partial Y(z)}{\partial \gamma} Y^{-1}(z)$  of its solution (cf. Lemma 2.1 of [7]):

$$(6.31) \quad X(z) \equiv \frac{\partial Y(z)}{\partial \gamma} Y^{-1}(z) = \frac{1}{2\pi i} \int_C Y_-(z') \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} Y_+^{-1}(z') \frac{\partial f(z')}{\partial \gamma} (z')^{-n} \frac{dz'}{z' - z} \\ = \frac{1}{2\pi i} \int_C \begin{pmatrix} -Y_{11}(z')Y_{21}(z') & Y_{11}^2(z') \\ -Y_{21}^2(z') & Y_{11}(z')Y_{21}(z') \end{pmatrix} \frac{\partial f(z')}{\partial \gamma} (z')^{-n} \frac{dz'}{z' - z}.$$

Now, from (6.1) and (6.16) we see that

$$\frac{dY_{11}(z)}{dz} = A_{11}(z)Y_{11}(z) + \Lambda_{11}^{-2}A_{12}(z)Y_{21}(z) - c(z)Y_{11}(z), \\ \frac{dY_{21}(z)}{dz} = \Lambda_{11}^2A_{21}(z)Y_{11}(z) + A_{22}(z)Y_{21}(z) - c(z)Y_{21}(z),$$

where  $c(z) = \sum_{k=0}^m \frac{\alpha_k}{z - z_k} + \frac{\lambda}{z}$ . This allows us to re-write (6.30) as follows:

$$\frac{\partial}{\partial \gamma} \ln D_n = \frac{1}{2\pi i} \int_C z^{-n} \left( Y_{11}^2(z) \Lambda_{11}^2 A_{21}(z) - \Lambda_{11}^{-2} A_{12}(z) Y_{21}^2(z) \right. \\ \left. + Y_{11}(z) Y_{21}(z) (A_{22}(z) - A_{11}(z)) \right) \frac{\partial f(z)}{\partial \gamma} dz \\ = \Lambda_{11}^2 \sum_{k=0}^m A_{k,21} \int_C Y_{11}^2(z) \frac{\partial f(z)}{\partial \gamma} \frac{z^{-n} dz}{2\pi i (z - z_k)} - \Lambda_{11}^{-2} \sum_{k=0}^m A_{k,12} \int_C Y_{21}^2(z) \frac{\partial f(z)}{\partial \gamma} \frac{z^{-n} dz}{2\pi i (z - z_k)} \\ + \sum_{k=0}^m (A_{k,22} - A_{k,11}) \int_C Y_{11}(z) Y_{21}(z) \frac{\partial f(z)}{\partial \gamma} \frac{z^{-n} dz}{2\pi i (z - z_k)} + \Lambda_{11}^2 B_{21} \int_C Y_{11}^2(z) \frac{\partial f(z)}{\partial \gamma} \frac{z^{-n} dz}{2\pi i z} \\ - \Lambda_{11}^{-2} B_{12} \int_C Y_{21}^2(z) \frac{\partial f(z)}{\partial \gamma} \frac{z^{-n} dz}{2\pi i z} + (B_{22} - B_{11}) \int_C Y_{11}(z) Y_{21}(z) \frac{\partial f(z)}{\partial \gamma} \frac{z^{-n} dz}{2\pi i z}.$$

A comparison with (6.31) yields the following *local* representation for the  $\gamma$ -derivative of  $\ln D_n$ :

$$(6.32) \quad \frac{\partial}{\partial \gamma} \ln D_n = \Lambda_{11}^2 \sum_{k=0}^m A_{k,21} X_{12}(z_k) + \Lambda_{11}^{-2} \sum_{k=0}^m A_{k,12} X_{21}(z_k) + \sum_{k=0}^m A_{k,11} X_{11}(z_k) + \sum_{k=0}^m A_{k,22} X_{22}(z_k) \\ + \Lambda_{11}^2 B_{21} X_{12}(0) + \Lambda_{11}^{-2} B_{12} X_{21}(0) + B_{11} X_{11}(0) + B_{22} X_{22}(0).$$

The last formula can be also written in the compact matrix form,

$$(6.33) \quad \frac{\partial}{\partial \gamma} \ln D_n = \sum_{k=0}^m \text{trace } \Lambda^{-1} A_k \Lambda X(z_k) + \text{trace } \Lambda^{-1} B \Lambda X(0).$$

By evaluating (6.33), with the help of the representation (6.5), one arrives at the formulae (6.28) and (6.29).  $\square$

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